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ENDNU MERE SPREDT FÆGTNING

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"ENDNU MERE SPREDT FÆGTNING" er en samling artikler fra min hånd skrevet siden samlingen "YDERLIGERE SPREDT FÆGTNING", IMFUFA-tekst nr. 470, 2010, der efterfulgte samlingerne "MERE SPREDT FÆGTNING", IMFUFA-tekst nr. 404, 2001, og "SPREDT FÆGTNING", IMFUFA-tekst nr.199, 1990.

Jeg har kaldt artikelsamlingen "ENDNU MERE SPREDT FÆGTNING", fordi der som for de foregående artikelsamlingers vedkommende umiddelbart er tale om spredt fægtning. Også her er artiklerne skrevet til forskellige lejligheder og til forskellige målgrupper. Artiklerne spreder sig fra grænselandet mellem didaktik/pædagogik og uddannelsespolitik og universitetspolitik til grænselandet mellem fysik og fysikdidaktik, svarende til mine engagementer i perioden. Udover for mig selv at tjene et ordenshensyn håber jeg, at sammenstillingen af artiklerne gør det muligt for andre end mig selv at øjne, at fægtningen trods alt ikke er mere spredt, end at en betydelig vedholdenhed gør sig gældende. En række KVANT artikler fra min hånd om breddeopgaver fra perioden er ikke medtaget.

Jens Højgaard Jensen, Juli 2019

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Ideen til den følgende artikel havde jeg i mange år liggende i skuffen, ventende på at blive udfoldet, når jeg fik tid. Det fik jeg efter at være gået af som studieleder for den naturvidenskabelige basisuddannelse på RUC i 2010. Artiklen blev først sendt til American Journal of Physics som replik på et eksempel bragt der. Men den blev afvist, hvorefter jeg sendte den til European Journal of Physics, hvor den blev optaget.

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Rules for rolling as a rotation about the instantaneous point of contact

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Abstract

It is a widespread misunderstanding in introductory physics courses that the motion of rolling bodies in general can be calculated using the point of contact as a reference point when equating the rate of change of angular momentum to the torque. In this paper I discuss in general two correct rules to be used instead, in order to derive the equation of motion of rolling bodies, taking moments about the point of contact. I also clarify that the point of contact either can be reckoned the fixed point on the rolling body instantly at the point of contact or the geometrical point defined as the point of contact at any time. Altogether this gives four different ways of deriving the equation of motion with some of them being easier than others depending on the case under consideration. The four different methods are as an illustration applied to a case presented recently by Turner and Turner (2010 *Am. J. Phys.* **78** 905–7).

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In my own introductory course in mechanics at university I remember being taught that in order to calculate the motion of rolling bodies the torque rule, $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$, may be applied about three different points.

1. A fixed point in an inertial frame of reference.
2. The centre of mass of the bodies.
3. The instantaneous point of contact in the rolling motion.

The torque rule is valid for any system of particles when taking moments either about a fixed point in an inertial frame of reference or about the centre of mass. The third possibility, taking moments about the point of contact, is specific for rolling motion and was justified by the general argument that the point of contact instantaneously is at rest and therefore can be reckoned a fixed point in the laboratory frame of reference.

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I later found this way of arguing widespread. For example, the *Berkeley Physics Course* argues in this way [2]:

> The acceleration in the motion of the rolling object is calculated by recognizing that *instantaneously* the motion is simply a rotation about a point on the periphery of the object. Thus we shall require the moment of force about P to equal the rate of change of angular momentum about P (P being the point of contact),

and Ohanian in this way [3]:

> The existence of an instantaneous fixed axis in rolling motion (without slipping) enables us to deal with this motion by methods developed in the preceding sections. The rotation about the instantaneous fixed axis obeys our old equation $I\mathrm{d}\omega/\mathrm{d}t = \tau_z$. From this, we can calculate the motion of a rolling body on which external forces and torques act. (z refers to a fixed axis in the laboratory system.)

But it is in general not correct to use the simple torque rule about the point of contact dealing with rolling motion. And the argument for it is not valid: a rolling motion can correctly be described as a rotation about the fixed point on the rolling body (or a fixed axis) instantaneously coinciding with the geometrically defined point of contact. From this description you can calculate the kinetic energy and the angular momentum about the point. But the rate of change of the angular momentum which equals the torque at the same instant involves a difference between the angular momentum at one time and the angular momentum about the same point at an infinitesimally different time. And that is not what is done when differentiating with respect to time the expression for the angular momentum about the (moving) point of contact.

In most introductory physics courses the case considered is a sphere or a cylinder rolling up or down an incline. In spite of the wrong justification the correct result is obtained in this case when using the simple torque rule about the point of contact. This may be the reason why the misunderstood justification is so widespread.

The main purpose of this paper is, besides pointing out the misunderstanding, to identify its possible replacements. How can we retain the convenience of taking moments about the point of contact (and therefore not needing to know the normal and frictional forces there) in the most general case?

In the next two sections I thus derive two general rules for the rate of change of the angular momentum about any moving point to replace the normal torque rule. The first rule is almost unknown in the literature, whereas the second rule is well known. I then clarify that the point of contact either can be reckoned the fixed point on the rolling body instantly at the point of contact or the geometrical point defined as the point of contact at any time. I have not seen this ambiguity discussed before.

Altogether this in general gives four different ways of deriving the equation of motion for a rolling body taking moments about the point of contact in order to avoid knowing the normal force and the frictional force acting on the body at the point of contact. Some of them are easier than others depending on the case in consideration. The four ways of attacking rolling as a rotation about the instantaneous point of contact are finally illustrated by applying them successively to rolling of a semicircular hoop as a case presented in a recent article [1], where Turner and Turner disprove the above-mentioned misunderstanding, their case being a counterexample.

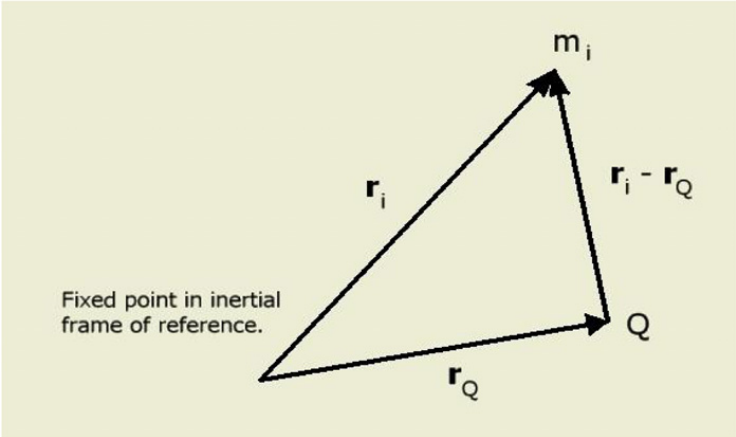

Figure 1. Position vectors relating to a moving point Q.

2. General rule, when moments are taken about an arbitrary moving point

When considering rolling as a rotation about the instantaneous point of contact as the *Berkeley Physics Course* and Ohanian actually do and how I was taught in my introductory course in mechanics, the angular momentum inserted in the torque rule in general terms is

$$\mathbf{L}_{QI} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \mathbf{v}_i; \qquad \mathbf{v}_i = \mathrm{d}\mathbf{r}_i/\mathrm{d}t, \tag{1}$$

where, as indicated in figure 1, $\mathbf{r}_i$ are the position vectors in an inertial frame of reference of the mass points m_i of the rolling body, the point of contact is called Q and $\mathbf{r}_Q$ is the position of this point in the inertial frame of reference. The index Q refers to the point about which moments are taken. The index I tells us that the velocities in the definition of $\mathbf{L}_{QI}$ are evaluated in an inertial frame of reference. In the textbooks L_{QI} is normally expressed as $I_Q\omega$. The logical mistake then appears when, before differentiating, I_Q is calculated with Q understood as the point of contact at any time, which means that $\mathbf{r}_Q$ therefore should be considered a function of time and not a fixed point as assumed when using the normal torque rule.

Considering that $\mathbf{r}_Q$ should be understood as a function of time, the quantity $\mathbf{L}_{QI}$ defined in equation (1) is a peculiar one. Viewed as an angular momentum, the velocities are measured relative to an inertial frame of reference and the position vectors relative to a frame of reference following Q. But that does not prevent us from asking how the normal torque rule should be changed in order to be consistent with the definition in equation (1). We will do so for an arbitrary system of particles.

As the point of departure we know the torque rule to be true when taking moments about a fixed point in an inertial frame of reference. If the resultant external force on particle i is called $\mathbf{F}_i$ we thus know

$$\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau} \quad \text{to be valid for} \quad \mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \quad \text{and} \quad \boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i. \tag{2}$$

Denoting the total mass of the system by M and the velocity of its centre of mass by $\mathbf{v}_{\mathrm{CM}}$, we then ask for the relation between

$$\mathbf{L}_{QI} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \mathbf{v}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i - \sum_i \mathbf{r}_Q \times m_i \mathbf{v}_i = \mathbf{L} - \mathbf{r}_Q \times M\mathbf{v}_{\mathrm{CM}} \tag{3}$$

and

$$\boldsymbol{\tau}_Q = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i - \sum_i \mathbf{r}_Q \times \mathbf{F}_i = \boldsymbol{\tau} - \mathbf{r}_Q \times \sum_i \mathbf{F}_i. \tag{4}$$

Differentiating equation (3) with respect to time gives us

$$\mathrm{d}\mathbf{L}_{QI}/\mathrm{d}t = \mathrm{d}\mathbf{L}/\mathrm{d}t - \mathbf{v}_Q \times M\mathbf{v}_{\mathrm{CM}} - \mathbf{r}_Q \times M\mathrm{d}\mathbf{v}_{\mathrm{CM}}/\mathrm{d}t. \tag{5}$$

Since $M\mathrm{d}\mathbf{v}_{\mathrm{CM}}/\mathrm{d}t = \sum_i \mathbf{F}_i$ according to Newton's second law for a system of particles and $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$, by combining equations (4) and (5), the answer is seen to be

$$\mathrm{d}\mathbf{L}_{QI}/\mathrm{d}t = \boldsymbol{\tau}_Q + M\mathbf{v}_{\mathrm{CM}} \times \mathbf{v}_Q. \tag{6}$$

This general equation is the correct one to be used in order to establish an equation of motion for rolling bodies taking as a starting point the angular momentum as understood in the citations from the *Berkeley Physics Course* and Ohanian among others, and as defined in equation (1).

The second term on the right-hand side vanishes if Q does not move or is identical to CM, as it should. However it also vanishes whenever $\mathbf{v}_{\mathrm{CM}}$ and $\mathbf{v}_Q$ are parallel vectors, which is the case if the rolling bodies are symmetric around an axis through CM perpendicular to the direction of rolling and if the rolling is taking place on a flat plane. As mentioned already, a possible reason for the misunderstood justification for using the torque rule $\mathrm{d}\mathbf{L}_{QI}/\mathrm{d}t = \boldsymbol{\tau}_Q$—contrary to equation (6)—in, e.g., the *Berkeley Physics Course* and Ohanian is therefore that one obtains the correct result in the cases being dealt with.

The derivation of equation (6) is straightforward. In spite of that, among textbooks in physics I only found it derived in two Danish ones [4, 5]. Perhaps the derivation of equation (6) is widespread in more technically oriented textbooks.

Most textbooks in physics do not give wrong arguments when dealing with rolling problems, since they avoid using moments about the point of contact. Some more advanced textbooks in mechanics such as [6, 7] derive another general rule than equation (6) also concerning moments about a moving point.

3. General rule, when moments are calculated in an accelerated frame of reference

Similar to equation (6) this general rule also refers to moments taken about an arbitrary moving point Q. But now the angular momentum, contrary to the definition given in equation (1), is defined with not only moments taken but also velocities evaluated relative to Q:

$$\mathbf{L}_{QQ} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i(\mathbf{v}_i - \mathbf{v}_Q). \tag{7}$$

The first index Q refers to the point about which moments are taken. The second index Q tells us that the velocities in the definition of $\mathbf{L}_{QQ}$ are evaluated in a frame of reference following Q.

Following the same procedure as in the last section when deriving equation (6) our point of departure is again the simple torque rule as expressed in (2). Also equation (4) for $\boldsymbol{\tau}_Q$ is maintained. But equation (3) has to be replaced by

$$\begin{aligned}\mathbf{L}_{QQ} &= \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i(\mathbf{v}_i - \mathbf{v}_Q) \\ &= \sum_i \mathbf{r}_i \times m_i\mathbf{v}_i - \sum_i \mathbf{r}_Q \times m_i\mathbf{v}_i - \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i\mathbf{v}_Q \\ &= \mathbf{L} - \mathbf{r}_Q \times M\mathbf{v}_{\mathrm{CM}} - (\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q) \times M\mathbf{v}_Q. \end{aligned} \tag{8}$$

Differentiating this gives

$$\begin{aligned}\mathrm{d}\mathbf{L}_{QQ}/\mathrm{d}t = \mathrm{d}\mathbf{L}/\mathrm{d}t - \mathbf{r}_Q \times M\mathrm{d}\mathbf{v}_{\mathrm{CM}}/\mathrm{d}t - \mathbf{v}_Q \times M\mathbf{v}_{\mathrm{CM}} - (\mathbf{v}_{\mathrm{CM}} - \mathbf{v}_Q) \times M\mathbf{v}_Q - (\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q) \\ \times M\mathrm{d}\mathbf{v}_Q/\mathrm{d}t. \end{aligned} \tag{9}$$

Since $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$ according to (2), $M\mathrm{d}\mathbf{v}_{\mathrm{CM}}/\mathrm{d}t = \sum_i \mathbf{F}_i$ according to Newton's second law for a system of particles, and the third and the fourth terms cancel each other, so by combining equations (4) and (9) the second general rule is then seen to be

$$\mathrm{d}\mathbf{L}_{QQ}/\mathrm{d}t = \boldsymbol{\tau}_Q + (\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q) \times (-M\mathrm{d}\mathbf{v}_Q/\mathrm{d}t). \tag{10}$$

In most textbooks this result, like here, is found by differentiating equation (7). But it could also be derived more directly: $\mathbf{L}_{QQ}$ as defined in equation (7) is the normal angular momentum about the moving point Q evaluated in a frame of reference following Q. In this frame of reference Q is a fixed point. Therefore $\mathrm{d}\mathbf{L}_{QQ}/\mathrm{d}t$ is equal to the sum of the torque about Q from the forces also existing in the laboratory frame of reference, contributing with $\boldsymbol{\tau}_Q$, and the torque about Q from the homogeneous field of fictitious forces $-M\mathrm{d}\mathbf{v}_Q/\mathrm{d}t$ is due to the acceleration of the frame of reference following Q.

As in equation (6), the second term vanishes if Q does not move or if Q is identical with CM, as it should. But besides these cases the two equations (6) and (10) are seen to have vanishing second terms under different circumstances due to the two different definitions of $\mathbf{L}_Q$ in equations (1) and (7). For, e.g., a symmetric body rolling on an inclined plane and taking Q as the geometrically defined point of contact, the second term in equation (6) is zero since $\mathbf{v}_{\mathrm{CM}}$ and $\mathbf{v}_Q$ are parallel, whereas the second term in equation (10) is different from zero when the body is accelerating.

4. Regarding rolling as a rotation about the instantaneous point of contact

Both equations (6) and (10) prove in general that it is incorrect to argue that the normal 'rate of change of angular momentum equals torque' rule without modification can be applied to rolling motion taking moments abound the instantaneous point of contact, just because this point is instantaneously at rest. In a recent article [1] Turner and Turner (T and T) demonstrated the incorrectness of the argument by calculating the equation of motion for a rocking semicircular loop as a counterexample. T and T used equation (10) referring to Q as the material point of the rolling body which is instantaneously coinciding with the point of contact. But since nothing was presupposed about Q when deriving equation (10), Q can equally well be taken as the point of contact understood as a moving geometrical point. Also in equation (6) Q can both be taken as the geometrically defined point of contact and the fixed point of the rolling body instantaneously at the point of contact. Since both $\mathbf{L}_Q$ and Q thus can be defined in two different ways, there are four different ways of regarding rolling as a rotation about the instantaneous point of contact. 'Rolling as a rotation about the instantaneous point of contact' is thus seen to be an ambiguous concept with a need for specification.

5. Rolling of a semicircular hoop

As an illustration of the need for specification we will successively apply the four ways of attacking rolling as a rotation about the instantaneous point of contact on the case of the semicircular hoop. The aim of this is not to improve the calculations of T and T concerning the motion of the semicircular hoop, and no new results are derived for this motion. The aim here is to demonstrate that the derivations of T and T are just one out of four possible ways of deriving the same results; one of the others may be easier. Since the message of the paper is the four ways and not the case of the semicircular hoop, the following four successive treatments of it are chosen to be rather compressed. Paper and a pencil are presumably needed in order to follow the arguments.

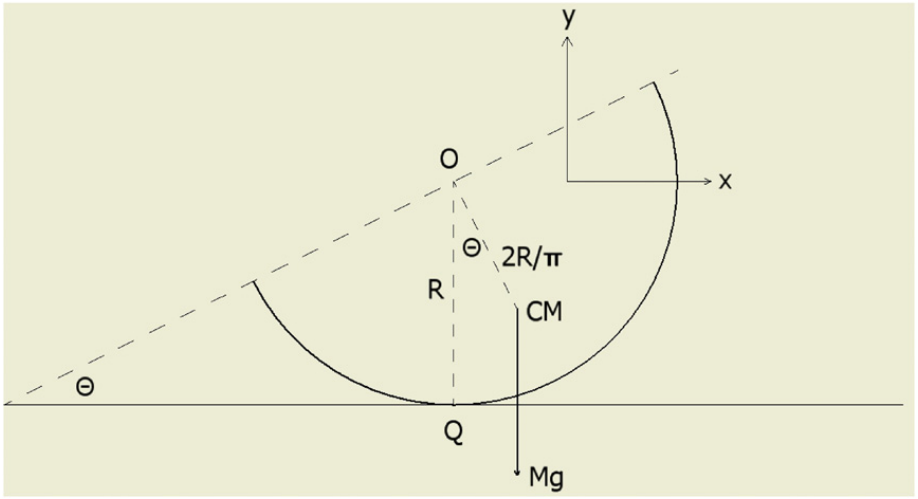

Figure 2. Rocking semicircular hoop.

Following T and T the hoop's point of contact Q is shown in figure 2, where the hoop is centred about point O, and its centre of mass is located at point CM, a distance $2R/\pi$ from point O along the perpendicular bisector of the hoop. The distance from O to Q is R. The force of gravity is Mg. The orientation and position of the hoop are characterized by the angle θ shown in the figure. The origin of the coordinate system in the figure is at the location of O, when $\theta = 0$. The direction of vectors perpendicular to the page are regarded as positive when pointing out of the page.

The torque about Q exerted by gravity is

$$\tau_Q = -(2\sin\theta/\pi)\cdot MgR. \tag{11}$$

The semicircular hoop's moment of inertia about Q is (see T and T for the detailed calculation)

$$I_Q = 2MR^2\cdot(1-2\cos\theta/\pi). \tag{12}$$

The mistake is now to equate τ_Q with $I_Q\cdot \mathrm{d}^2\theta/\mathrm{d}t^2$, thereby obtaining

$$(1-2\cos\theta/\pi)\cdot \mathrm{d}^2\theta/\mathrm{d}t^2 = -(\sin\theta/\pi)\cdot(g/R) \tag{13}$$

for the equation of motion for the rocking hoop.

Contrary to this, T and T expressed the frictional and normal forces acting on the semicircular hoop at Q in terms of the horizontal and vertical accelerations of the centre of mass of the hoop, calculated the torques about the centre of mass of the two forces, put the sum of these equal to $I_{\mathrm{CM}}\cdot \mathrm{d}^2\theta/\mathrm{d}t^2$, and thereby obtained the correct equation of motion for the rocking semicircular hoop:

$$(1-2\cos\theta/\pi)\cdot \mathrm{d}^2\theta/\mathrm{d}t^2 = -(\sin\theta/\pi)\cdot(g/R) - (\sin\theta/\pi)\cdot(\mathrm{d}\theta/\mathrm{d}t)^2. \tag{14}$$

Comparing equations (14) and (13) T and T then asked for the origin of the extra term on the right-hand side of equation (14), when taking moments about Q and not CM. In the following the answer, as mentioned, will be given in four different ways, where one of them is the one given by T and T.

(1) Q, a fixed point on the hoop; velocities evaluated in a frame of reference following Q
To take this way we use equation (10) as our point of departure. Here $\mathrm{d}\mathbf{L}_{QQ}/\mathrm{d}t$ is seen to be equal to I_Q multiplied with $\mathrm{d}^2\theta/\mathrm{d}t^2$, since I_Q does not depend on time, Q being a fixed point on the hoop. From figure 2 it is seen that

$$\mathbf{r}_{\mathrm{CM}} = (2\sin\theta/\pi - \theta, -2\cos\theta/\pi) \cdot R, \tag{15}$$

$$\mathbf{r}_Q = (-\theta, -1) \cdot R, \tag{16}$$

and therefore

$$\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q = (2\sin\theta/\pi,\ 1 - 2\cos\theta/\pi) \cdot R. \tag{17}$$

The velocity of Q (not to be confused with $\mathrm{d}\mathbf{r}_Q/\mathrm{d}t$, which is the velocity of the geometrically defined point of contact) is zero at the instant when Q coincides with the point of contact. The mistake is then to miss that this does not mean that the acceleration of Q is zero at that instant. In contrast it can be found to be (see T and T for the detailed calculation)

$$\mathrm{d}\mathbf{v}_Q/\mathrm{d}t = (0, (\mathrm{d}\theta/\mathrm{d}t)^2) \cdot R. \tag{18}$$

Inserting equations (11), (12), (17) and (18) into equation (10) one obtains equation (14) and not equation (13). This is the answer given by T and T to their question.

(2) Q, a fixed point on the hoop; velocities evaluated in the laboratory frame of reference
To take this way we use equation (6) as our point of departure. Here the second term on the right-hand side is zero, because $\mathbf{v}_Q$ is zero, Q being the fixed point on the hoop coinciding with the point of contact. But in order to evaluate $\mathrm{d}\mathbf{L}_{QI}/\mathrm{d}t$ we have to take into account that $\mathbf{L}_{QI}$ is only instantaneously equal to $I_Q \cdot \mathrm{d}\theta/\mathrm{d}t$. At all other instants the rotational contribution, $I_Q \cdot \mathrm{d}\theta/\mathrm{d}t$, to L_{QI} must be added a contribution arising from the homogeneous velocity field $\mathbf{v}_Q$:

$$(\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q) \times M\mathbf{v}_Q. \tag{19}$$

Differentiating the contribution (19) to $\mathbf{L}_{QI}$ we thus (using $\mathbf{v}_Q = 0$) obtain the additional term $(\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q) \times \mathrm{M}\mathrm{d}\mathbf{v}_Q/\mathrm{d}t$ to $\mathrm{d}\mathbf{L}_{QI}/\mathrm{d}t$ in addition to the contribution from the rotation about Q. The additional term is seen to be the same as that in equation (10) with opposite sign and on the other side of the sign of equation. Since I_Q is independent of time also in this case we therefore end up with the same calculations as in the former case. But now the additional term should not be termed 'phantom torque', as T and T call it, but rather 'phantom change of angular momentum'. I would prefer not to use any of the names, since nothing mysterious is happening.

(3) Q, the geometrically defined point of contact; velocities evaluated in the laboratory frame of reference
We again use equation (6), because the velocities are evaluated in the laboratory frame of reference. In this case $\mathrm{L}_{QI} = I_Q \cdot \mathrm{d}\theta/\mathrm{d}t$ all the time, since the motion of the semicircular hoop here at any instant can be described solely as a rotation about Q. But now I_Q varies with time. With input from equation (12) the left-hand side of equation (6) in this case is therefore

$$\begin{aligned}\mathrm{dL}_{QI}/\mathrm{d}t &= \mathrm{d}I_Q/\mathrm{d}t \cdot \mathrm{d}\theta/\mathrm{d}t + I_Q \cdot \mathrm{d}^2\theta/\mathrm{d}t^2 \\ &= 2MR^2 \cdot (2\sin\theta/\pi) \cdot (\mathrm{d}\theta/\mathrm{d}t)^2 + 2MR^2 \cdot (1 - 2\cos\theta/\pi) \cdot \mathrm{d}^2\theta/\mathrm{d}t^2.\end{aligned} \tag{20}$$

Differentiating equation (15) gives

$$\mathbf{v}_{\mathrm{CM}} = (2\cos\theta/\pi - 1,\ 2\sin\theta/\pi) \cdot R \cdot \mathrm{d}\theta/\mathrm{d}t. \tag{21}$$

In the same way $\mathbf{v}_Q$ is given here by differentiating equation (16):

$$\mathbf{v}_Q = (-\mathrm{d}\theta/\mathrm{d}t,\ 0) \cdot R. \tag{22}$$

For the term $M\mathbf{v}_{\mathrm{CM}} \times \mathbf{v}_Q$ on the right-hand side of equation (6) we therefore get

$$MR^2 \cdot (2\sin\theta/\pi) \cdot (\mathrm{d}\theta/\mathrm{d}t)^2, \text{ pointing out of the page, when it is positive.} \tag{23}$$

Inserting equations (20), (11) and (23) in equation (6) is then again seen to lead to equation (14) as the equation of motion for the rocking semicircular hoop. This is perhaps the simplest way to derive the equation of motion.

(4) Q, the geometrically defined point of contact; velocities evaluated in a frame of reference following Q

We make use of equation (10), because the velocities are evaluated in a frame of reference following Q. Contrary to equation (18) $\mathrm{d}\mathbf{v}_Q/\mathrm{d}t$ is now given by differentiating equation (22). Combining the result of that with equation (17) we then obtain for the term $(\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q) \times (-M\mathrm{d}\mathbf{v}_Q/\mathrm{d}t)$ on the right-hand side of equation (10)

$$-MR^2 \cdot (1 - 2\cos\theta/\pi) \cdot \mathrm{d}^2\theta/\mathrm{d}t^2, \text{ pointing out of the page, when it is positive.} \tag{24}$$

Since the motion seen from the laboratory frame of reference all the time can be described solely as a rotation about Q, the motion seen from the frame of reference following Q can be described as this rotation minus a translation with velocity $\mathbf{v}_Q$. Besides the contribution $I_Q \cdot \mathrm{d}\theta/\mathrm{d}t$ to $\mathbf{L}_{QQ}$ from the rotation the contribution $-M(\mathbf{r}_{\mathrm{CM}} - \mathbf{r}_Q) \times \mathbf{v}_Q$ due to translation thus has to be added. Using equations (12), (17) and (22) we obtain

$$\begin{aligned} \mathrm{L}_{QQ} &= 2MR^2 \cdot (1 - 2\cos\theta/\pi) \cdot \mathrm{d}\theta/\mathrm{d}t - MR^2 \cdot (1 - 2\cos\theta/\pi) \cdot \mathrm{d}\theta/\mathrm{d}t \\ &= \tfrac{1}{2} I_Q \cdot \mathrm{d}\theta/\mathrm{d}t. \end{aligned} \tag{25}$$

$\mathrm{dL}_{QQ}/\mathrm{d}t$ is in this case half the value of that derived in equation (20). Inserting this, the value of τ_Q from equation (11) and the value given by (24) for the second term on the right-hand side of equation (10) again leads to equation (14) as the equation of motion for the rocking semicircular hoop.

6. Conclusion

In general the motion of rolling bodies cannot be calculated using the point of contact as a reference point when setting the rate of change of angular momentum equal to the resulting moment of the forces operating in the laboratory frame of reference. It is possible to avoid knowing the normal and frictional forces at the point of contact by taking moments about the contact point. But in general the usual law must then be modified. If velocities are evaluated in the laboratory frame of reference equation (6) is the modified version of the usual law. If velocities are evaluated in a frame of reference following the point of contact equation (10) is the modified version of the usual law. When using these two general laws it is necessary to clarify whether the point of contact is reckoned the fixed point on the rolling body instantly in the point of contact or the geometrical point defined as the point of contact at any time. Altogether this gives four different ways of deriving the equation of motion with some of them being easier than others, depending on the case under consideration.

The rules for rolling as a rotation about the instantaneous point of contact presented here can be used as an example to demonstrate that textbooks even in physics are not always correct. But the example can also be used to demonstrate that in a subject like classical mechanics mistakes can be straightened out by critical analysis, going back to the basic definitions and assumptions. In this way, the example provides a case for illuminating a characteristic of physics: arguments can be critically tested, even by non-experts.

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Den følgende artikel har været trykt i "RUCnyt - 08 – 2012", RUCs daværende meddelelses og debatblad. Anledningen var truende besparelseskrav til NAT området. "Igen-igen" i titlen refererer til, at det ikke var første gang i historien, at NAT på RUC følte sig presset økonomisk. I et HUM/SAM domineret universitet uden medicin eller ingeniørvidenskab, altså andre "våde" fagområder, som NAT kunne støtte sig til, har det altid været en særlig udfordring på RUC at skabe forståelse for omkostningerne ved at praktisere "våde" fag.

Debat

NATURVIDENSKAB PÅ RUC

NAT – igen, igen!

Af Jens Højgaard Jensen, tidligere dekan for det naturvidenskabelige hovedområde, tidligere prorektor for RUC og tidligere studieleder for den naturvidenskabelige basisuddannelse.

Af orienteringen i nyhedsbrevet fra det sidste bestyrelsesmøde den 15. december 2011, fremgår det, at RUC´ s bestyrelse på mødet vedtog rektoratets indstilling til RUC´ s budget for 2012. Dog udestår der stadig indgåelse af en leveranceaftale med NSM. Hvorfor bestyrelsen ønsker NSM` s budget sat på dagsordenen igen på et af de første møder i 2012 med rektoratets og instituttets indstilling til, hvorledes 2,4 mio. kr. (netto) kan findes inden for NSM` s budget, idet dette forudsættes af bestyrelsen.

Der er således konflikt omkring finansieringen af NAT på RUC. RUC erfarne vil sige: IGEN! RUC veteraner vil sige IGEN, IGEN! Jeg vil her forsøge, at forklare, hvad de igen- kommende konflikter bunder i og antyde, hvad der skal til for at overkomme dem. Det vil ikke blive særlig teknisk, da jeg ikke mere har tilstrækkeligt overblik over tallene til det. Men jeg synes alligevel, jeg kan bidrage ved at påpege mønstre, jeg kender gennem alle årene på RUC.

Naturvidenskab speciel på RUC

RUC er det eneste danske universitet, som har et naturvidenskabeligt hovedområde uden samtidigt også at have ingeniørvidenskab og/eller lægevidenskab. Det betyder, at naturvidenskab er specielt på RUC ved i forhold til det øvrige RUC at være alene om at skulle forklare, hvad der skal til for at undervise og forske i "våde" fag på niveau med konkurrenterne på de andre universiteter. Det er ikke nogen nem sag, hverken at forstå eller forklare. Det er tilfældet, selvom realiteten er, at vilkårene for NAT på RUC sammenholdt med det øvrige "tørre" RUC er noget dårligere end på de andre danske universiteter og universiteter i det meste af verden i det hele taget.

For at få orden på diskussionen er det vigtigt, at man gør sig klart, at RUC som andre universiteter i Danmark har de tre indtægtskilder af samme størrelse: forskningskontraktpenge, uddannelsestaxameterpenge og basisbevillingen på finansloven. I det følgende vil jeg lade forskningskontraktpengene ude af betragtning, da de på NAT- området i hovedsagen går til løn til ekstra personale og ekstra udstyr og derfor mere eller mindre afføder udgifter svarende til indtægterne fra dem. Tilbage er at diskutere sammenhængen imellem udgifter og indtægter for så vidt angår uddannelsestaxameterpengene og basisbevillingen.

I de seneste 15-20 år har søgningen til naturvidenskab på RUC som ved de øvrige universiteter i Danmark og den vestlige verden været faldende. Det ser imidlertid ud til, at vi nu både på RUC og andre steder er på vej opad i søgning igen. Antallet af ansøgninger til nat-bas har været stigende de sidste to år. Der vil imidlertid gå nogen tid før nat-området igen har så mange studerende, at det ikke på taxametersiden er en underskudsforretning. I perioden er antallet af lærere reduceret på grund af det faldende studentertal. F.eks. er antallet af fastansatte lærere i både matematik og fysik reduceret fra 12 til 7. Men da det f. eks. kræver næsten den samme lærerindsats at levere kursusundervisning for 10 studerende som for 30 studerende har det ikke været muligt at sænke lærerbestanden proportionalt med faldet i studenterbestanden. Derfor er nat-området taxametermæssigt en underskudsforretning og vil også være det nogle år frem i tiden. Men forhåbentlig mindre og mindre.

Problematisk udregning

Imidlertid er det "underskud" for først og fremmest NSM, der siden 2006 er opereret med på RUC noget helt andet og meget mere dramatisk, end det, der kan henføres til mangelen på studerende. Og det hænger sammen med den måde finanslovens basisbevilling til RUC indgår i RUC´ s budgettering på.

For en snes år siden og tilbage i tiden var universiteternes basisbevillinger på finansloven opdelt efter fakulteterne, således at de relative andele i RUC´ s tilfælde begrundet med henholdsvis humaniora, samfundsvidenskab og naturvidenskab fremgik. Billedligt talt kan man først jævnføre udgifter til tegneundervisning og svømmeundervisning, hvis udgifterne til svømmehallen holdes uden for jævnførelsen. Og andelen af basisbevillingen begrundet i naturvidenskab i forhold til humaniora og samfundsvidenskab afspejlede noget sådant. RUC´ s muligheder i fremtiden for at få løftet vores for lille basisbevilling sammenlignet med basisbevillingerne ved de store universiteter vil også være afhængig af i hvilken grad, der kan henvises til naturvidenskabelige behov.

Men i RUC´ s nuværende måde at udregne institutternes "udgifter", "indtægter" og deraf udregnede "underskud" optræder kun en lille andel af finanslovens basisbevilling som indtægt. Størstedelen af de bogførte udgifter er lønningerne til de ansatte i de forskellige institutter ved opstillingen af de enkelte institutters budgetter, og størstedelen af de bogførte indtægter er taxameterpengene fra den leverede undervisning. Medens en rimelig andel af finanslovens basisbevilling begrundet ikke mindst i institutternes aktiviteter som sagt ikke bogføres som en indtægt. Det bogførte "underskud" er alene forskellen mellem lønninger og taxameterindtægter. Finanslovens basisbevilling er en indtægt for RUC som helhed, der bruges til bygninger, fællesadministrationen, biblioteket m. m. Men ikke til at afholde infrastrukturomkostningerne til at naturvidenskab kan fungere "vådt". Hvis der på RUC skal dyrkes naturvidenskab i samme forstand som på andre danske universiteter er NSM derfor dømt til at være født med et "underskud", så længe det defineres, som det gøres, uanset hvor mange studerende der uddannes. For de naturvidenskabelige dele i CEBIT og ENSPAC gælder tilsvarende problemer, her blot placeret på de indre institutbaner.

Bistandshjælp eller nødvendige sigtepunkter?

For at dække NSM´ s "underskud", sådan som det er defineret, tildeles instituttet en såkaldt "ramme", der af det øvrige RUC dårligt kan opfattes som andet end en slags bistandshjælp. Det medfører to store problemer: For det øvrige RUC virker NSM som et bundløst kar. Hvornår hører henvisningen til de særlige behov for "våde" fag op? For de ansatte på NSM svinder motivationen, da det ikke er muligt at rette blikket imod en fremtid uden

"underskud". Hvornår giver en ekstra indsats gevinst, som ikke inddrages?

Det har ikke altid været sådan. Før 2006 blev forholdet imellem de enkelte institutter og RUC som helhed i det væsentlige reguleret ved hjælp af personalerammer. Personalet i fællesadministrationen, biblioteket *og* Tek-TAP på NAT var budgetteret konkret efter kvalitative vurderinger. Herefter var rammerne for VIP-stillingerne og de decentrale Adm-TAP stillinger gjort studentertalsafhængige. Jeg mener at huske, at NSM havde 6 VIP-stillinger flere end berettiget til ifølge denne model i 2006 svarende til ca. 3 millioner kr. i underskud. Modsat "underskuddet", som det defineres nu, der var på 20 millioner kr. i 2006. Som det ses var mulighederne for at afvikle det taxameterindtægtsbegrundede underskud på 3 millioner helt anderledes end at imødegå "underskuddet" på 20 millioner på grund af den manglende andel af finanslovens basisbevilling til dækning af grundudgifterne til "våd" naturvidenskab.

Hvis naturvidenskab skal trives på RUC og RUC trives med naturvidenskab er det nødvendigt, at der også nu gives begrundede basisbevillinger til naturvidenskabsmiljøerne, blandt andet med henvisning til basisbevillingen på finansloven og forholdene ved de konkurrerende universiteter, forud for den del af bevillingen, der gøres taxameterstyret. Grundudgifterne skal ikke dækkes af en "ramme", der af RUC i al almindelighed opfattes som bistandshjælp. Kun derved kan der genetableres nødvendige sigtepunkter fælles for RUC og naturvidenskab.

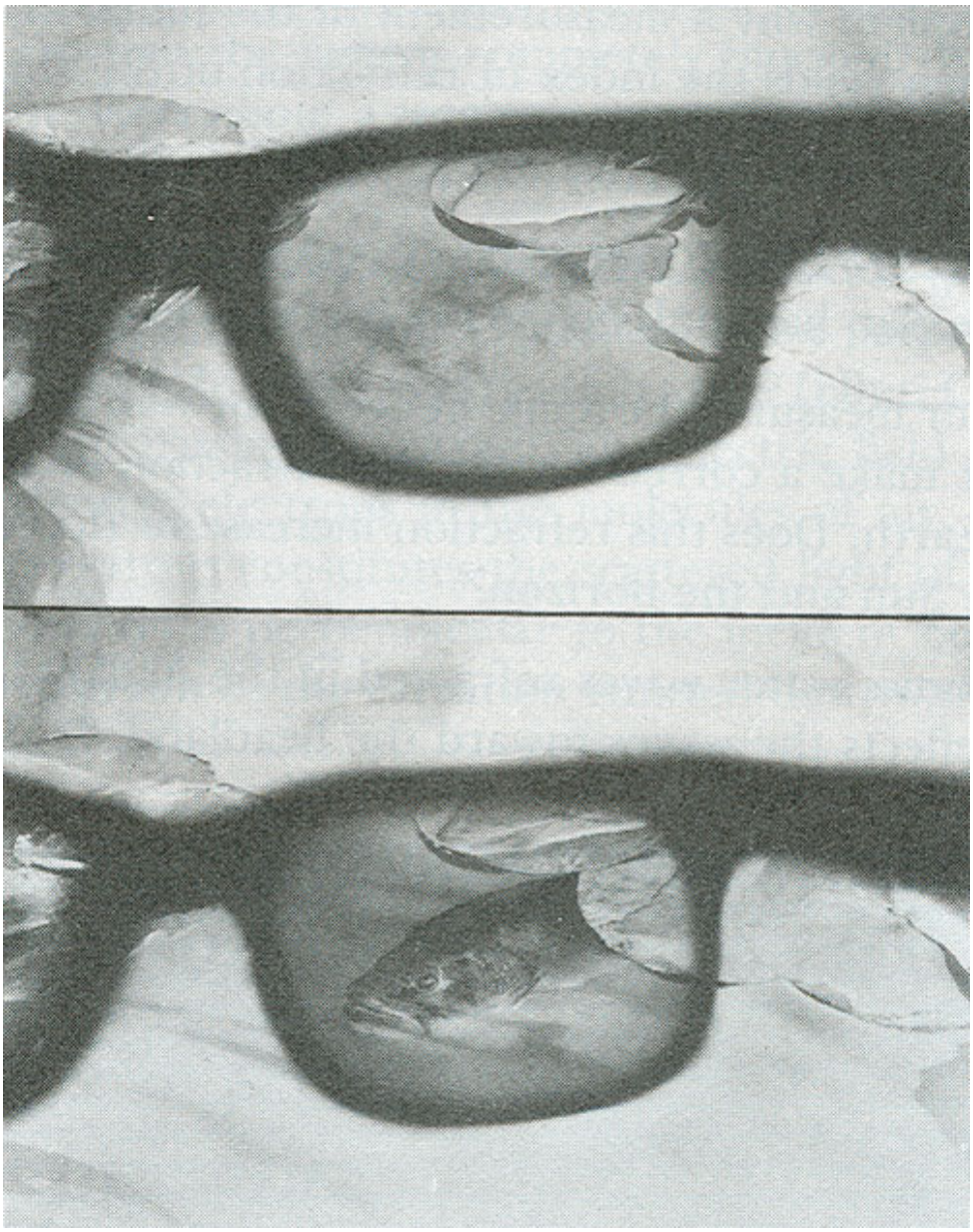


Fig. 37.32 When viewed through ordinary sunglasses (top), the fish is almost completely hidden by the glare reflected by the water. When viewed through Polaroid sunglasses (bottom), the fish becomes visible because the glare is reduced.

“Det sete afhænger af øjnene, der ser”

Den følgende artikel er skrevet på opfordring til bogen "Tværvidenskab i teori og praksis", Hans Reitzels Forlag, 2012, redigeret af Thyge Enevoldsen og Erling Jelsøe. Målgruppen var først og fremmest samfundsvidenskabelige bachelorstuderende på RUC. Opfordringen glædede mig.

2

På tværs af videnskaberne

Jens Højgaard Jensen

Indledning

"Hvorfor behandler I de samme emner så forskelligt i undervisningen i kemi og i fysik? Kan I ikke samarbejde lidt bedre om det?", spurgte Trine, som studerer kemi og fysik i kombination på RUC. Og det udløste et større foredrag fra min side.

Jeg sagde til Trine, at hun havde ret i, at der er emner inden for f.eks. termodynamik og kvantefysik, der er fælles for kemi og fysik. Og, at jeg ikke ville sige, at der ikke godt kunne koordineres noget bedre, end der bliver gjort. Men, at hun skulle være opmærksom på, at kemi og fysik på trods af overlappende emner er ret så forskelligartede fag. Fysik har sit udgangspunkt i naturfilosofien, som det f.eks. kan ses af titlen på Newtons hovedværk fra 1687 og startskuddet for fysik i vore dages forstand: *Philosophiae Naturalis Principia Mathematica* ("De matematiske principper for naturfilosofien"). Hvorimod kemi ved siden af alkymien har rod i praktiske problemer inden for landbrug, minedrift og medicin. Og disse forskelligartede forhistorier præger stadig de to fag og deres udøvere.

Fysikere er opdraget til at tænke problemindsnævrende og rette sig imod sagens kerne: Start enkelt. Vælg den simplest mulige model, og regn først på *den.* Fanger den essensen? Hvis ikke, så komplicer gradvist, indtil essensen er fanget. Men komplicer ikke mere end nødvendigt. Det er den typiske måde for fysikere at nærme sig et problem på. Og tænkemåden kan få fysikere til at virke som nogen, der bilder sig ind, at de i princippet forstår alle mulige ting, som de ikke i praksis har fod på. På samme måde som filosoffer åbenbart må have virket på Niels Bohr, der ellers var en høflig mand, da han omtalte dem som nogen, der ville forstå alting uden at lære noget. Men faktisk har fysikeres måde med gennem matematisk modellering at nærme sig problemer vist sig nyttig ikke blot til udvikling af fysikken selv. Undertiden har fysikeres måde at tænke på også vist sig produktiv inden for kemi, datalogi, molekylærbiologi, geologi, geografi,

økonomi, finansiering, trafikregulering og mange slags ingeniørfag. Og tænkemåden har jo rødder langt tilbage i historien. I Platons hulemetafor eksempelvis genfinder vi fysikerens tænkemåde: Stands ikke ved at iagttage skyggerne af personerne på væggen modsat huleåbningen, som du sidder med ryggen til. Skyggerne er blot fremtrædelser. Essensen er personerne. Og det er personerne, du skal nå frem til en erkendelse af gennem dine iagttagelser af skyggerne. At fysik i nyere tid er kommet til at stå som faget, der i udpræget grad inkarnerer essensorienteringen ses f.eks. af Karl Marx' forord til hans hovedværk *Kapitalen* fra 1867. Han siger blandt andet heri, at nogen har fremført, at hans analyse af det kapitalistiske samfund ikke rækker til at beskrive datidens Tyskland og alene er en beskrivelse af datidens England. Og til dem siger han, at han opererer som fysikeren, der "iagttager naturprocesser, enten der, hvor de fremtræder i deres mest prægnante form og er mindst forplumret af forstyrrende indflydelser, eller han foretager, hvor det er muligt, eksperimenter under betingelser, som sikrer processens rene forløb." (Marx, 1970. B.1: 30). Som Galilei, der undersøger det frie fald ved at måle på faldet af en sten og ikke et blad. Og herved finder frem til essensen i det frie fald, som også er styrende for bladets fald uanset luftmodstandens modifikationer. På samme måde er studiet af datidens England med den fuldt udfoldede kapitalisme det, der kan føre frem til forståelsen af de drivende kræfter bag også Tysklands udvikling.

Kemikere er derimod opdragede til, at deres tænkning skal virke i praksis. De vil ikke nøjes med at forstå ting i princippet. Billedligt talt oplever kemikere hjælpen fra fysikken som den hjælp en taxachauffør får ved at blive gjort opmærksom på, at gadenettet i byer typisk enten er som et edderkoppespind eller som et kvadreret papir. At være klar over, om den by, der køres i, er af den ene eller anden slags, er selvfølgelig en hjælp for en taxachauffør. Men der er langt herfra og til at finde rundt. På samme måde er fysikkens essentielle love normalt langt fra tilstrækkelige til at løse kemikernes konkrete problemer. Kemi må derfor udvikle sine egne mere anvendelsesorienterede love og opskrifter. Det grundfaglige og det anvendelsesorienterede flyder mere sammen i kemi end i fysik. Det ses f.eks. ved opslag af stillinger inden for kemi. Farmaceuter, civilingeniører i kemi og cand.scient'er i kemi kommer her normalt i betragtning på lige fod. Derimod er det sjældent, at der opslås stillinger, hvor cand.scient.'er i fysik og maskin-, bygnings- eller elektro-civilingeniører er stillet lige. Faktisk har modstillingen af grundfagene matematik og fysik på den ene side og de anvendte maskin-, bygnings- og elektro- fag på den anden side altid væ-

ret en afgørende del af de tekniske universiteters udviklingshistorie. Hvor grundfagene interesserer sig for generelle delaspekter ved fænomenerne, er de anvendte fags interesse fænomenerne som konkrete helheder. På samme måde er kemikere mere forpligtet på konkrete helheder, medens fysikere i højere grad er på jagt efter delaspekter, der lader sig generalisere.

Oplevelsen af forskellige meldinger om de samme emner fra lærere i kemi og lærere i fysik, skal frem for at bidrage til frustration betragtes som et vigtigt bidrag til uddannelsen. Det kan godt være, at en del kunne hentes ved bedre koordinering. Men det er også indiskutabelt sådan, at fagkundskab ikke kun handler om emnekundskab. *Fag handler også om de måder, emnerne anskues på*. Og her adskiller kemi og fysik sig altså – ikke kun på RUC, men overalt i verden. At studere kemi og fysik parallelt giver netop muligheden for at forstå, hvordan dagsordener i kemi og fysik er forskellige, og derved få nemmere ved at manøvrere i sammensatte miljøer, end hvis man først og fremmest helliger sig det ene af fagene.

Sådan cirka lød "foredraget". Jeg havde svært ved at stoppe, da Trine havde sat mig i gang. Sådan er det tit med lærere, når emnet har vores interesse. Men til min glæde havde jeg ikke snakket Trine flad. Tværtimod. Hun var helt oplivet og med på mit budskab. Men hun ærgrede sig over, at hun ikke havde fået åbnet den synsvinkel på sine studier før. Hun havde ikke forud tænkt på sine fag som anskuelsesmåder snarere end som emneområder. Men hun kunne godt genkende billedet, jeg havde forsøgt at tegne. Og hun ville have været befriet for frustrationer, hvis det var blevet ridset op for hende tidligere i hendes studieforløb.

Pointen i at fortælle anekdoten om min og Trines snak som optakt til et kapitel med titlen "På tværs af videnskaberne" kan måske anes. Den vil i alle tilfælde blive uddybet i det følgende. Men først vil jeg præsentere mig selv. Jeg er uddannet fysiker fra 1970 og blev ansat på RUC fra starten i 1972. Også før min ansættelse på RUC var jeg engageret i uddannelses- og universitetspolitik. Men som én fra første ansættelsesbølge på et nyt universitet under etablering blev uddannelses- og universitetspolitik siden en dominerende del af mit VIP-liv på RUC. Tilbage i 70`erne var jeg dekan for det naturvidenskabelige hovedområde, i perioden 1986-1992 var jeg prorektor for RUC og i perioden 1997-2010 studieleder for den naturvidenskabelige basisuddannelse. Jeg har deltaget i etableringen af den naturvidenskabelige basisuddannelse, fysikuddannelsen, det tidligere institut for matematik og fysik (IMFUFA), og kombinationsuddannelsesstrukturen på RUC. Samtidig har jeg undervist på skift på den naturvidenskabelige basisuddannelse og på overbygningen i fysik og udviklet

nye kurser til begge dele. Forskningsmæssigt er jeg skiftet fra fysik til naturvidenskabsdidaktik.

Min baggrund gør, at jeg først og fremmest ser diskussioner om *tværvidenskab* som en uddannelses- og forskningspolitisk sag.. Som fysiker, naturvidenskabsdidaktiker og bidragyder til RUC's udvikling i bredere almindelighed er det for mig slående, hvor forskelligt fag er konstitueret og følgelig, hvor afhængig meningen af en tværfaglighedsdiskussion derfor er af sammenhængen. Jeg kan illustrere med mig selv som eksempel: Som fysiker – hvilket for mig betyder en person med en bestemt tillært optik at anskue verden i kraft af – oplever jeg megen tværfaglighedsretorik praktisistisk grænsende til det populistiske. Som naturvidenskabsdidaktiker – hvilket betyder en person med et refleksionsprojekt rettet imod praktisk undervisningsforbedring – gælder kvalmen omvendt akademisk oppustethed i branchen uden forpligtethed på sagen. Kaare Pedersen har karakteriseret ambitionen for samfundsvidenskabeligt projektarbejde som "saglig faglighed" i værktøjsbogen om "Problemorienteret projektarbejde", som han og Poul Bitch Olsen har skrevet til primært samfundsvidenskabelige studerende(Olsen & Pedersen, 1997: 17). Jeg er på linje med ham, når det gælder naturvidenskabsdidaktik. For at være samfundsrelevant skal naturvidenskabsdidaktik dyrkes tværfagligt og sagligt praksisrettet. Derimod ligger den langsigtede samfundsrelevans i fysik oftest som et biprodukt af grundfaglig sandhedssøgen, hvor anvendelsesorienterede tværfaglighedskrav opleves som forfladigende.

For mig ligger det samfundsmæssigt mest perspektivrige i kravet om tværfaglighed generelt set i, at det kan trække i retning bort fra fagidioti og overspecialisering imod en mere kosmopolitisk faglighed. RUC`s kombinationsstruktur inviterer f.eks. hertil. Hvis de to valgte fag har en tilstrækkelig afstand mellem sig, kan det akademiske faglandskab ses med større overblik, end hvis det kun anskues ud fra en enkelt faglig bakketop. Og det giver erfaring med, at det sete afhænger af fra, hvilken bakketop der ses.

Men jeg ser også betydningen af tværfaglighed som et krav om at underordne fag en sag. Det er f.eks. det, der er brug for i naturvidenskabsdidaktik.

Det følgende er et forsøg på at give et fugleperspektiv på tværfaglighedsdiskussionen på universiteterne i det hele taget. Sådan som jeg ser den. Til det hører først at forholde sig til, hvad de fag eller videnskaber, der skal gås på tværs af, er for nogle størrelser. Det vil ske i afsnittet "Fag". Dernæst vil jeg give mit bud på nogle væsensforskellige betydninger af, hvad der menes med tværfaglighed, i afsnittet "Tværfaglighed". På baggrund heraf

vil jeg i afsnittet "De to kulturer" kort referere og kommentere traditionelle opfattelser af kulturkløften imellem de humanistiske videnskaber i bred forstand og naturvidenskaberne i bred forstand. Det fører mig så frem til i afsnittet "Universiteternes fem fagkulturer" at skitsere min opfattelse af, hvor de dybeste kløfter i universitetslandskabet befinder sig. Endelig forsøger jeg – ud fra de foregående afsnit – i det afsluttende afsnit om "Tværfaglighed og samfundsvidenskab" at placere tværfaglighedsdiskussionen inden for samfundsvidenskab i den større universitetssammenhæng.

Fag

Ordet fag dækker over mange betydninger. Der er meget, der kaldes fag. Og hvad tværfaglighed er og kan gøre godt for afhænger derfor af, hvad det er for en slags fag, der skal gås på tværs af. Lad os derfor til en start orientere os lidt i virvaret.

Ifølge Ordbog over Det Danske Sprog har ordet fag følgende to hovedbetydninger:

1. "afgrænset flade ell. rum, især hvor flere følger i række efter hinanden; afdelt område."
2. "afdeling ell. afgrænset område inden for menneskelig **viden**, **virksomhed** osv."

Når vi bruger ordet fag på universiteterne refererer det til mindst fire forskellige slags afsæt:

1. **Vidensfag**, f.eks. historie eller matematik.

Historie og matematik har eksisteret som en "afdeling af menneskelig viden" lang tid før, der har eksisteret historikere eller matematikere. Nu om dage kan det være en profession at være historiker eller matematiker. Men afsættet var, at der eksisterede afdelinger af menneskelig viden forud, som skulle forvaltes og videreudvikles.

2. **Professionsfag**, f.eks. kok, læge eller tømrer.

Her er afsættet forskellige afgrænsede områder af menneskelig virksomhed, som den samfundsmæssige arbejdsdeling med tiden har udgrænset

til at skulle varetages af fagfolk. Det har så affødt behov for faguddannelser og siden f.eks. både ernæringsforskning, lægevidenskab og forskning i bygningskonstruktioner.

3. **Skolefag**, f.eks. historie, matematik, husgerning, sløjd, engelsk eller oldtidskundskab.

Ved skolens organisering i fag sker afgrænsningerne både med henvisning til vidensområder, som i tilfældene historie og matematik, og med henvisning til virksomhedsområder, som i tilfældene sløjd og husgerning. Men skolen konstruerer også sine egne fag som f.eks. oldtidskundskab og sammenstykningen af kultur- litteratur- og sprogelementer til et fag som engelsk. Da universiteterne bl.a. skal uddanne lærere til skolen er også skolens indretning et af afsættene for universiteternes fagopdelinger.

4. **Problemfag**, f.eks. miljøplanlægning, internationale udviklingsstudier eller sprog- og kulturmødestudier.

Universiteterne udvikler også nye fag med afsæt i nye samfundsmæssige problemfelter. Med tiden kan det føre til udviklingen af nye professioner, som f.eks. i tilfældet datalogi. TEK-SAM, IU og sprog- og kulturmødestudier er eksempler på sådanne fag udviklet specielt på RUC.

Opdelingen af fag i vidensfag, professionsfag, skolefag og problemfag har med fagenes oprindelse at gøre. Var faget et vidensområde, forud for det blev en profession at være forvalter af det pågældende vidensområde? Eller var faget i udgangspunktet et virksomhedsområde, som udviklede tilknyttet viden? Under alle omstændigheder er fag noget, der ændrer og udvikler sig med tiden. Det gælder også vekselvirkningen imellem videns- og virksomhedsaspekterne af givne fag.

Men ved siden af at spørge til fagenes forskelligartede oprindelse, kan man også spørge til karakteren af deres indhold, uafhængigt af hvordan fagene kom til verden. Her skelnes der traditionelt mellem **anvendte fag** og **grundfag** – i alle tilfælde for mig som fysiker – en vigtig skelnen, selvom den ikke benyttes så selvfølgeligt, som det tidligere har været tilfældet. Skematisk kan anvendte fag og grundfag, som jeg ser det, modstilles således:

	ANVENDTE FAG	GRUNDFAG
Forskellige ambitioner	Relevans	Sandhed
Forskellige reduktioner	Specifikke helheder	Generelle delaspekter

Hvad angår anvendte fags og grundfags forskellige ambitioner skal skemaet naturligvis ikke forstås sådan, at anvendte fag er usande og grundfag irrelevante.

Relevanspres kan ganske vist føre til usandhed. Den såkaldte Lysenko-affære i Sovjetunionen i 1948 og efterfølgende år er et dramatisk eksempel på det. Landbrugsspecialisten Trofin Lysenko havde i trediverne udviklet sin egen teori om evolution og nedarvning, hvor han i modsætning til den etablerede genetik mente, at tillærte egenskaber kan nedarves. De praktiske muligheder for forbedringer i landbruget, hvis Lysenkos teori var rigtig, fik Stalin til at vedtage teorien som rigtig i en grad, så modstandere af teorien blev udrenset, fængslet og i nogle tilfælde dræbt. Men det hører også med til billedet, at fiaskoerne af nogle af Lysenkos praktiske projekter, i begyndelsen af tresserne førte til, at genetikken som videnskabelig disciplin blev genindført i Sovjetunionen. I længden var den anvendte landbrugsvidenskab bedre tjent med at basere sig på sandheder end på ønsketænkning.

Selv i en så løst struktureret anvendt faglighed som naturvidenskabsdidaktik, hvor evidensen for praktisk succes eller fiasko er svagere end i landbrugsvidenskab, satser vi også på, at ærlighed varer længst.

Grundfaglig isolation i elfenbenstårnene fører ganske vist i mange tilfælde til sande ligegyldigheder, men historien er også fuld af eksempler på, at helt afgørende samfundsudviklinger udsprang fra netop grundfag. Et klassisk eksempel er fysikernes arbejde i 1800-tallet med at forstå sammenhængen imellem elektricitet og magnetisme, som endte op med Cambridge-fysikeren James Clerk Maxwells matematiske sammenfatning af sagen i 1873 i de såkaldte Maxwells ligninger. Ligningerne viste sig ikke blot at være en teori for elektricitet og magnetisme, men også en teori for lys som elektromagnetiske bølger. Og de forudsagde muligheden af radiobølger som elektromagnetiske bølger med længere bølgelængder end lys. 15 år efter blev radiobølgerne påvist første gang af Hertz. De praktiske konsekvenser af menneskers beherskelse af radiobølger til kommunikationsformål har siden vist sig kolossale. Hvis Maxwell i stedet for at dyrke grundforskning på Cambridge havde været ansat af f.eks. American Telephone and Telegraph Company til at forbedre datidens luftbårne te-

lekommunikation var det næppe ham, der var kommet frem til Maxwells ligninger. I stedet kunne han tænkes at have arbejdet med at forbedre de hulspejle, som man dengang morsede over afstand med.

Sammenfattende kan det måske siges sådan, at for de anvendte fag er relevansen ideelt set målet og sandheden midlet. Medens det for grundfagene ideelt set er sandheden, som undertiden viser sig at have relevans, der er målet uden hensyn til relevansen.

Som illustration af, hvor stærkt forskellen imellem anvendte fag og grundfag kan opleves, vil jeg kort berette om forskellige hold af naturvidenskabelige basisstuderendes oplevelser på DTU for en del år siden. Som led i et undervisningsforløb om "halvlederfysik og halvlederteknologi i Københavnsområdet" besøgte vi bl.a. et fysikinstitut og et elektronikinstitut på DTU. Forud for besøgene opfordrede jeg de studerende til diskret at spørge til, hvad de mente om fysikerne på elektronikinstituttet og, hvad de mente om elektronikerne på fysikinstituttet. Det slog aldrig fejl, at de fik reaktioner i retning af: Elektronikerne fandt fysikerne umådeligt arrogante. De mente i princippet at have svar på alting uden at være i stand til at anvende deres viden i praksis. Omvendt mente fysikerne, at elektronikerne nærmere hørte hjemme på Teknologisk Institut end på et universitet. Lidt karikeret, men genkendeligt for de involverede, tror jeg.

Kløften imellem elektronikerne og fysikerne har med relevansambition versus sandhedsambition at gøre. Men mere afgørende er måske de forskellige reduktioner, der præger de to fagkulturer. Både for elektronikerne og for fysikerne er f.eks. Maxwells ligninger centrale. Men selvom ligningerne umiddelbart ser ens ud på tavlen hos elektronikerne og fysikerne er det ikke det samme de to fagkulturer "ser" i dem. For elektronikerne er Maxwells ligninger et redskab blandt andre til at udvikle f.eks. mikrofoner. For fysikerne er mikrofoner en illustration blandt andre af induktionsloven som en del af Maxwells ligninger. Elektronikerne fokuserer på mikrofonen som konkret helhed. Den skal virke. Fysikerne fokuserer på induktionsloven som et delaspekt ved mikrofonen, som har generel interesse som naturlov.

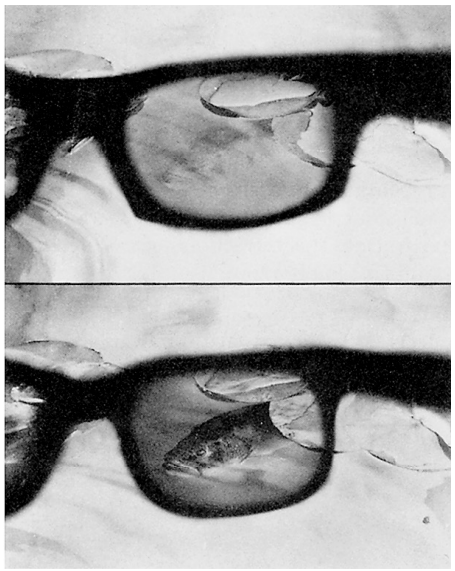
Jeg har nu skitseret, hvordan fag kan have forskellig slags oprindelse og forskellig slags indhold. Jeg kunne også have forsøgt at kortlægge deres forskellige samfundsfunktioner. Fagene kunne også have været karakteriseret ud fra om de var domineret af et eller sammensat af flere paradigmer. Ligeledes kunne fagene være beskrevet som af mere eller mindre konstruktivt skabende eller analytisk udredende art. Eller i forhold til, hvorvidt de fortrinsvis er af empirisk eller teoretisk art.

Det afgørende budskab, som jeg har villet fremhæve i afsnittet her er, at fag, uanset om man afgrænser dem fra hinanden sociologisk eller erkendelsesmæssigt, er meget forskelligartede. Derfor er tværfaglighed også en meget forskelligartet øvelse afhængig af hvilken slags fag, der skal gås på tværs af.

Tværfaglighed

Ud over at være afhængig af, hvilke slags fag det drejer sig om, er spørgsmålet om tværfaglighed også afhængig af, hvad formålet med tværfagligheden er. Man kan ikke svare på, hvordan man skal arbejde tværfagligt, hvis man ikke er klar over, hvorfor tværfagligheden efterspørges. Jeg vil tydeliggøre det ved hjælp af et eksempel, der angår fagene geologi og fysik.

Både geologi og fysik er fag, hvis kerner, modsat så mange andre fag, kan karakteriseres ved hver sin karakteristiske anskuelsesmåde. Hvad jeg mener med "anskuelsesmåde", kan billedligt illustreres ved hjælp af figur 1.2.



Figur 1.2 *At se er at se bort fra. Øverst er guldfisken set gennem almindelige solbriller næsten usynlig på grund af genskinnet fra vandoverfladen. Nederst ses guldfisken tydeligt gennem polaroid solbriller, fordi polariseringen af brillerne medfører, at den forstyrrende del af lyset frasorteres.*

Kilde H. C. Ohanian, Physics, sec. ed. exp., W.W.Norton & Company, New York-London, 1989, side 915.

At have tilegnet sig en anskuelsesmåde er som at have lært at anskue verden gennem et særligt polaroid filter. Geologernes og fysikernes filtre er

helt forskellige. Billedlig talt er de trænet i at anvende helt forskellige polariseringsretninger. Men geologerne og fysikerne er fælles om, at deres faglighed er forbundet tæt til bestemte filtre. Geologerne ser ikke bjergformationer som os andre. De læser et historisk forløb ind i, hvad de ser. Bjergene, som de foreligger, fungerer for dem, som kilderne gør for historikere. Geologer er uddannet som *naturhistorikere*. På samme måde ser fysikere ikke bevægelser som andre. De ser dem som manifestationer af altomfattende matematiske bevægelseslove. Som for filosoffer tjener enkeltfænomener som afsæt til generaliseringer. Fysikere er uddannet som *matematiske essenstænkere*. Der er ikke nødvendigvis tale om, at geologerne overser, at bjerge kan opfattes som andet end historiske levn. Der er heller ikke nødvendigvis tale om, at fysikerne overser, at bevægelse kan opfattes som andet end et matematisk mønster. Både geologerne og fysikerne ser imidlertid bort fra andet end det, de er uddannet til at fokusere på. Men *at se bort fra* er ikke det samme som *at overse*. Ved at se bort fra, for dem forstyrrende fænomener og forhold, er de stand til på hver deres måde at se "guldfisk", som når der på figur 1.2 ses gennem polaroid-solbrillerne.

Min far, som var professor i fysik ved Københavns Universitet, var for noget mere end en snes år siden formand for et udvalg, som havde til opgave at give forslag til, hvordan behovet for dansk ekspertise i olieefterforskning kunne imødekommes studieplansmæssigt ved Københavns Universitet. Da olieefterforskning både indebærer viden om undergrunden og viden om metoderne til at undersøge undergrunden med, bestod udvalget af netop både geologer og fysikere. Geologerne og fysikerne i udvalget var overbevist om nødvendigheden af hinandens ekspertise. Men da det kom til stykket, viste det sig (frit gengivet efter min far), at geologerne anerkendte fysik som et nyttigt redskab til visse praktiske formål, hvorimod geologi var en tilegnet anskuelsesmåde a la figur 1, som gør det muligt for den kvalificerede at få øje på forhold, som ikke kan iagttages af den ukvalificerede. Med fysikerne forholdt det sig omvendt. Derfor fandt geologerne, at en stort set normal geologiuddannelse suppleret med et mindre kursus i fortolkning af seismiske data var den rigtige grunduddannelse af olieefterforskningseksperter. Hvorimod fysikerne fandt, at en stort set normal fysikuddannelse suppleret med et mindre kursus til at blive fortrolig med rækkefølgen af de forskellige lag i undergrunden var den rigtige grunduddannelse af olieefterforskningseksperter. Det var min fars indtryk, at denne situation ikke var speciel for Danmark. Olieindustrien led under den i hele verden.

Pointen med denne anekdote er ikke fagchauvinismen, men det, der nærer den. Fagfolk ved normalt godt, at de ser bort fra ting, når de tager deres polaroid-briller på. Men de overser typisk, at der findes briller med andre polariseringsretninger end deres egne. Andre fag end ens eget anskues mere som en samling konkrete fakta, færdigheder og opskrifter end som alternative anskuelsesmåder. Man har jo typisk aldrig prøvet at se verden igennem de andres briller.

I takt med fagenes øgende specialisering øges vanskeligheden ved at leve sig ind i, hvad de andres fag går ud på. Og det har omkostninger, f.eks. for olieselskaberne, når geologer og fysikere får sværere og sværere ved at tale fornuftigt sammen. Omkostningerne ved den øgede specialisering af fagene er derfor en af årsagerne til efterspørgslen efter tværfaglighed. Tværfaglighed efterspørges med det formål at bygge bro mellem fagkulturer. I f.eks. olieefterforskningstilfældet er det ikke en direkte anvendelsesrettet uddannelse vendt mod olieefterforskning, som savnes på universiteterne (den tager firmaerne sig af). Det er nærmere uddannelsen af de studerende og deres lærere til både at kunne tænke som en geolog og en fysiker på samme tid, der efterspørges. Og det er det, universiteternes organisering efter specialiseringer stiller sig hindrende i vejen for.Så vidt eksemplet angående geologi og fysik. Jeg håber, at det illustrerer én af drivkræfterne bag efterspørgslen efter tværfaglighed i universitetsuddannelserne siden 60`erne. Den stigende specialisering af forskningen og de videregående uddannelser siden Anden Verdenskrig har ført til tab af overblik og gjort samarbejde vanskeligere.

En hel anden slags efterspørgsel efter tværfaglighed hænger sammen med problem- og funktionsorientering af forskning og uddannelser. For eksempel i naturvidenskabsdidaktik og store dele af samfundsvidenskaberne. Her drejer det sig ikke om at skabe dialog imellem forskellige fagkulturer, men nærmere om at nedbryde kunstige faggrænser, der står i vejen for, at problemer og funktioner kan adresseres seriøst. Eller med Kaare Pedersens ord: *sagligt.* Sagen skal forpligtige og fagene underordnes sagen. Tværfaglighed er så ikke et mål i sig selv men en følge af problem- og funktionsorienteringen.

Ordet tværfaglighed benyttes således i helt forskellige betydninger og af helt forskellige grunde. Ud over de to nævnte benyttes ordet også ved bestræbelser på at udvikle et nyt fag på grænsen imellem to eksisterende fag. De tre helt forskellige brug af ordet tværfaglighed med dertil hørende forskellige intentioner, som jeg kan få øje på, ser for mig skematisk sådan ud:

TRE SLAGS TVÆRFAGLIGHED		
Fagintegration på grænseområdet mellem to fag	**Fagintegration bestemt af et praktisk problem- og funktionsfelt**	**Forståelse og overblik på tværs af fag**
F.eks.: Matematisk økonomi mellem matematik og økonomi; biokemi mellem biologi og kemi; geofysik mellem geologi og fysik.	F.eks.: Erhvervsøkonomi, der integrerer økonomi, sociologi, jura osv.; lægevidenskab, der integrerer kemi, fysiologi, psykologi osv.; ingeniørvidenskab, der integrerer matematik, fysik, geologi osv.	F.eks.: RUC's kombinationsuddannelsesstruktur
Dvs.: Tværfaglighed = specialisering	Dvs.: Tværfaglighed = integration af grundfagselementer til et anvendt fag	Dvs.: Tværfaglighed = at være faglig kosmopolit
Tværfagligheden efterspørges: for at understøtte etableringen af nye specialiserede fag.	Tværfagligheden efterspørges: for at fremme anvendeligheden af grundfag i sammenhæng med praktisk problemløsning.	Tværfagligheden efterspørges: for at imødegå uheldige konsekvenser af fagspecialisering.

I almindelighed betyder kosmopolit en verdensborger, dvs. en person, der føler sig hjemme i hele verden på tværs af landegrænser. Her har jeg brugt ordet i forhold til at føle sig hjemme på tværs af faggrænser. Parallellen mellem lande og fag er ikke dårlig. Fag er ligesom lande historiske konstruktioner. Men det betaler sig ikke at se bort fra eksistensen af fag og faglige kulturforskelle, hvis man vil arbejde på tværs. På samme måde som det ikke betaler sig som verdensborger at se bort fra eksistensen af forskellige nationalkulturer.

I sidste afsnit om fag var hovedbudskabet, at der er mange måder at være fag på (som der er mange måder at være land på), og at tværfaglighed derfor er en meget forskelligartet øvelse afhængig af, hvilken slags fag, der er tale om. I dette afsnit er hovedbudskabet, at tværfaglighed efterspørges af meget forskellige grunde og med meget forskellige hensigter. Både tværfaglighedsambitioner i retning af grænseområdefag og anvendte fag er af gammel dato. Og der er også brug for dem nu. Men personligt finder jeg det mest påtrængende at underbygge parolen om tværfaglighed med initiativer til at imødegå uheldige konsekvenser af fagspecialiseringen.

Under alle omstændigheder er det vigtigt ikke at spørge "hvordan tværfaglighed?" før spørgsmålet "hvorfor tværfaglighed?" er overvejet. Tværfaglighed er både et plusord og et gummiord. Derfor er man, som altid, når "gummiplusord" er i spil, bedst tjent med at være opmærksom på, hvilket formål brugen af ordet tjener i sammenhængen.

"De to kulturer"

I 1959 holdt C. P. Snow en forelæsning på universitetet i Cambridge med titlen "De to kulturer". Senere blev foredraget trykt, oversat til mange sprog og diskuteret i en sådan grad, at udtrykket "de to kulturer" siden er blevet gængs i mange sammenhænge verden over.

For at forstå sammenhængen til diskussionen af tværfaglighed skal man være opmærksom på Snows brug af ordet "kultur". På samme måde som f.eks. antropologer bruger han ordet til at betegne en gruppe af personer, "der lever i de samme omgivelser og er forenet gennem fælles vaner, fælles forudsætninger og en fælles livsform".(Snow C. P. 1966: 56). Man kan også sammenholde betydningen af ordet "kultur" i udtrykket "de to kulturer" med fysikeren og videnskabsteoretikeren Thomas S. Kuhns senere begreb om paradigme, som er det fælles sæt af videnskabelige værdier, antagelser om virkeligheden., metoder og nøgleeksempler, der får et videnskabeligt samfund til at hænge sammen.

Fire år efter sin forelæsning replicerede Snow skriftligt på ros og ris i den voldsomme debat den affødte. Her opridser han forelæsningens væsentlige indhold således:

> I vort samfund (det vil sige de industrialiserede vestlige samfund) har vi mistet enhver antydning af en fælles kultur. Mennesker der er uddannet med den største grundighed man kan forestille sig, er blevet ude af stand til at ommunikerer med hinanden om deres største intellektuelle problemer. Dette har alvorlige følger for vort skabende og intellektuelle liv – og frem for alt for hverdagslivet. Det gør det vanskeligt eller umuligt for os at handle rigtigt. Jeg gav det grelleste eksempel på denne manglende kommunikation ved at omtale to grupper mennesker, der repræsenterer hvad jeg har døbt "de to kulturer". Den ene gruppe omfatter naturvidenskabsmændene, hvis betydning, indsats og indflydelse det er overflødigt at understrege. Den anden omfatter humanisterne. Jeg mente ikke at det er humanisterne der træffer alle vigtige beslutninger i den vestlige verden. Jeg mente at humanisterne repræsenterer, udtrykker og til en vis grad former og forudsiger den ikke–naturvidenskabe-

> lige kulturs klima. De træffer ikke beslutningerne, men deres udtalelser og meninger har indflydelse på dem der gør det. Mellem disse to grupper – naturvidenskabsmændene og humanisterne – er der næsten ingen kommunikation, og i stedet for fællesfølelse nærmest noget i retning af fjendtlighed.(…)
>
> Som afslutning på dette resumé skal det siges at det naturligvis er umuligt at finde en fuldstændig løsning. Med de forhold vi lever under i dag – og i hvilken som helst tidsalder vi kan forudse – er renæssance – mennesket en umulighed. Men noget kan vi gøre. Vor største mulighed ligger i uddannelsen, først og fremmest når det gælder folkeskolen og gymnasiet, men også colleges og universiteter. Der er ingen undskyldning for at lade næste generation blive lige så totalt uvidende eller blottet for forståelse og sympati som vi selv er." (Snow, C. P. 1966: 53)

I forhold til de tre slags tværfaglighed og de tre slags grunde hertil skitseret i sidste afsnit ses Snows ærinde at svare til den tredje slags. Samfundet har behov for større forståelse og overblik på tværs af fag og flere faglige kosmopolitter. Og det største problem er her brobygningen over kulturkløften mellem naturvidenskab og humaniora. Snow havde selv et meget personligt forhold hertil. Han var uddannet fysiker for siden at blive skønlitterær forfatter og omgikkes folk fra begge kulturer. I hans venstreorienterede optik var humanisterne navlepillende maskinstormere mens naturvidenskabsfolkene var kulturelt isolerede, men i sidste ende var det deres indsats, der sikrede en bedre verden. Især var det et problem, at det politiske niveau først og fremmest var præget af den humanistiske kultur og uden kontakt til den naturvidenskabelige. Snows synspunkter var således ikke alene præget af hans dobbelthed som fysiker/forfatter, men også af, at der i tiden omkring 1959 – uanset, at forståelsen af dem var forbeholdt en snæver kreds – bredt i samfundet og politisk en meget stærk tro på de grundfaglige naturvidenskabers samfundsbetydning.

Piet Hein, der – ligesom Snow – både havde studeret fysik og var forfatter og derfor også havde et ben i hver side af kulturkløften, kaldte uafhængigt af Snow, men samtidigt med ham dem på humaniorasiden af kløften for "kultister" og dem på naturvidenskabssiden for "teknoter". At ordet kultist minder om "okkultist", og ordet teknot om "idiot" var for Piet Hein pointen i hans navngivning. Debatten om de to kulturer lå således i tiden. Og den var ret følelsesladet. Snows forelæsning er i sig selv – også efter hans egen mening – ikke særlig epokegørende. Det interessante er mere den megen debat den affødte dengang. Og at udtrykket "de to kul-

turer" er blevet hængende som noget, der refereres til over hele verden siden. I 1960'erne var det datidens voldsomme eksplosion i søgningen til de naturvidenskabelige uddannelser, som affødte snakken om "de to kulturer". Nu om dage er det helt modsat rekrutteringsproblemerne til naturvidenskab, der forklares med henvisning til "de to kulturer". Der er heller ikke samme optimistiske syn på naturvidenskab nu som i 60'erne. Men uanset hvad der skal forklares, er "de to kulturer" tilsyneladende en optik, der er blevet stående. Hvorfor er det sådan?

Jeg tror,at en del af forklaringen på, at snakken om "de to kulturer" appellerer stærkt til mange af os, er fordi der er så mange almenmenneskelige fortællinger forbundet med den. Begrebsmodstillinger som natur/kultur, abstrakt/konkret, erfaring/spekulation, objektiv/subjektiv, teoretisk/praktisk, kvantitet/kvalitet og opsplitning/helhed ligger lige om hjørnet. Og om det næste hjørne ligger begrebspar som udvikling/harmoni, fornuft/følelse, arbejde/familie og mand/kvinde.

En noget snævrere forklaring findes hos økonomen og den senere nobelpristager Paul Krugman (1996). I artiklen "Anti-økonomiske intellektuelle" harcelerer han over, at aviser og magasiner frembærer de forkerte akademiske pointer. En del af grunden til det er efter hans mening, at litterært orienterede intellektuelle er fjendtligt indstillede over for økonomer. Ifølge Krugman er fjendtligheden ikke så meget politisk, som den er kulturel: "Min pointe er i forvejen velkendt: at vores samfund endnu er splittet mellem C. P. Snows to kulturer. Som sagt på en anden måde, foregår der en stadig kamp mellem *to opfattelser af, hvad det vil sige at være intellektuel.* Den ene kultur er humanistisk og litterær, den anden er matematisk og videnskabelig." (Krugman 1996: 14-15) Fjendtligheden skyldes i forbindelse med f.eks. økonomiske synspunkter på international handel efter Krugmans mening, "at *de litterære ikke bryder sig om* den gængse økonomiske handelsanalyse, fordi den nødvendigvis får udtryk på et sprog, de hverken forstår eller har lyst til at forstå. For den som er fortrolig med selv den mest grundlæggende matematiske tænkning, kan den klassiske analyse af international handel, sådan som den blev udviklet af John Stuart Mill, forekomme nærmest banal i sin enkelhed. Men at formidle analysen uden matematikken – at redegøre for den grundlæggende historie udelukkende på et letfatteligt dagligsprog – er næsten umuligt, hvad syv generationer af økonomer også kan bevidne." (Krugman 1996: 14-15)

Selvom økonomi, ligesom historie eller sociologi omhandler mennesket og dets handlinger, står økonomi som et fag, hvis ideer normalt udtrykkes i form af matematiske modeller, ifølge Krugman solidt placeret

på den naturvidenskabelige side af kulturkløften. Kløften er for Krugman ikke imellem naturvidenskab og humaniora, men "mellem dem, der føler sig hjemme med en mere eller mindre matematisk tilgang til verden, og dem, der ikke gør det." (Krugman 1996: 14-15)

I bogen *Stregen i sandet, bølgen på vandet*" (Hansen 2000). er geologen Jens Morten Hansen også inde på matematikken som den store deler. I modsætning til Krugman er budskabet imidlertid ikke, at humanisterne burde tilegne sig mere matematisk tankegang. Hans ærinde er tværtimod at imødegå det matematisk-fysiske paradigmes dominans af geologi, der har pågået siden Galilei. Hvor fag som fysik og kemi er logiske i deres orientering, er fag som geologi og biologi historiske. Og det er forkert at underlægge de naturhistoriske fag et matematisk – logisk fagparadigme. For Jens Morten Hansen findes kulturkløften mellem matematik, fysik og kemi på den ene side og de naturhistoriske fag sammen med humaniora og samfundsvidenskab som også er historiske fag på den anden side.

Jeg er enig med Paul Krugman og Jens Morten Hansen i, at adgangen til matematisk tankegang eller dominansen fra den er afgørende i forståelsen af kløften mellem "de to kulturer". Men der er også andre årsager til kløften. Og først og fremmest er der også andre – måske ikke helt så dybe, men dybe nok endda – kløfter i det videnskabelige landskab. I kortlægningen af universiteternes fagkulturer kan man standse ved Snows to. Men det er efter min mening for grov en kortlægningen af de kulturkløfter, som det er særligt udfordrende for tværfagligheden at slå bro over. Kortet kan detaljeres, men på den anden side skal kortet heller ikke tegnes så detaljeret, at man ikke kan "se skoven for bare træer". Efter bedste evne har jeg ud fra mine egne erfaringer forsøgt mig med en opdeling af universitetslandskabet i fem fagkulturer, som jeg finder sigende og samtidigt til at overskue.

Universiteternes fem fagkulturer

Når vi på RUC for os selv og andre skal forklare de tværfaglige pointer i vores kombinationsstruktur, sker det på to ret forskelligartede måder, der hver især svarer til ret så forskelligartede opfattelser af, hvorfor tværfaglighed er at foretrække. Den ene forklaring – svarende til tværfaglighed i betydning 2 i det ovenstående skema – går ud fra tværfaglighed som fagintegration rettet imod et praktisk problem- eller funktionsfelt. Fidusen ved kombinationsstrukturen er så, at den på en ubureaukratisk måde inviterer til fagsamarbejder og ajourføring af fagsamarbejderne alene i kraft af initiativer fra involverede lærere og studerende.

Den anden forklaring – svarende til tværfaglighed i betydning 3 i skemaet – går ud fra gensidig fagperspektivering som en afgørende pointe ved tværfagligt samarbejde, hvor kombinationsstrukturens dobbeltfaglighed som noget særligt for RUC er en invitation til at blive uddannet som faglig kosmopolit med to forskellige faglige fædrelande.

I forhold til ambitionen om at uddanne faglige kosmopolitter i kraft af kombinationsstrukturens dobbeltfaglighed er kulturafstanden imellem de to valgte fag afgørende. At være barn af en østriger og en tysker giver større spændvidde end at være barn af en sjællænder og en jyde. Men at være barn af en tysker og en franskmand giver selvfølgelig en afgørende større forudsætning for at kunne spænde over europæiske kulturkløfter. Når RUC opererer med ca.25 fagkulturer svarende til RUC' s ca.25 overbygningsfag kan man sammenligne det med EU' s ca.25 medlemsstater. Men hvad angår kulturkløfter er det mere interessant at tænke på Europa som f.eks. præget af sammenstød imellem en empirisk og pragmatisk orienteret angelsaksisk kultur, en idealistisk og teoretisk orienteret germansk kultur og en formelt og rationalistisk orienteret romansk kultur. På tilsvarende måde kan vi efterspørge de dybereliggende kulturkløfter i universitetslandskabet.

Nedenstående opstilling viser mit forsøg på en kvalitativ beskrivelse af universiteternes og hermed RUC' s faggeografi ved fem fagkulturer.

UNIVERSITETERNES 5 FAGKULTURER

- **Den matematisk modellerende fagkultur**

Den matematisk modellerende fagkultur eksisterer på RUC først og fremmest i IMFUFA (matematik og fysik-delen af NSM), hvor den er dominerende. Derudover eksisterer den på RUC som minoritetskultur i mindre enklaver i kemi, datalogi og økonomi. På RUC spiller kulturen omfangsmæssigt ikke den store rolle, som den f.eks. gør det på Danmarks Tekniske Universitet (DTU).

- **Den empirisk eksperimenterende fagkultur**

Denne kultur er på RUC først og fremmest hjemmehørende i biologi- og kemidelene af NSM og miljøbiologi-delen af ENSPAC, hvor den er dominerende. Derudover findes den minoritetskultur på IMFUFA, geografi og Tek-Sam-delene af ENSPAC. Uden for RUC er den f.eks. den dominerende fagkultur på (den tidligere) Landbohøjskolen (KVL).

- **Den dialektisk reflekterende fagkultur**

Det er den klassiske universitetskultur, som på RUC særligt præger CUID. Men kulturen rækker også indover det øvrige humaniora, samfundsvidenskab og IMFUFA. Kulturens primære hjemsted uden for RUC er de humanistiske fakulteter på de traditionelle universiteter.

- **Den sagsorienterede analyserende fagkultur**

Kulturen er på RUC dominerende på ISG, ENSPAC, PAES og erhvervsøkonomi påCBIT. Det er i det hele taget den dominerende fagkultur på RUC med baggrund i RUCs ambitioner om have samfundsrelevans. Handelshøjskolen (CBS) er et eksempel på en institution uden for RUC, der domineres af kulturen.

- **Den konstruktivt skabende fagkultur**

På RUC er kulturen samlet på CBIT med kommunikation, journalistik, datalogi og performancedesign. Hvor de øvrige fire fagkulturer er analytiske i deres karakter, orienterer kulturen sig nærmere imod konstruktion end analyse. Et eksempel på en institution uden for RUC, hvor kulturen dominerer, kunne være Arkitektskolen.

Jeg er godt klar over, at skitsen er grov og slet ikke kan bruges til klassifikation i almindelighed. En teoretisk fysiker kan med rimelighed placeres i den matematisk modellerende fagkultur. En eksperimentalfysiker befinder sig på én gang i den matematisk modellerende fagkultur og i den empirisk eksperimenterende fagkultur. Men hvad med en elektroingeniør? Her er der ikke blot disse to, men også den konstruktivt skabende fagkultur og den sagsorienterede analyserende fagkultur i spil. Eller hvad med en læge? Her er både den empirisk eksperimenterende fagkultur og den sagsorienterede analyserende fagkultur på banen. Skal en jurist placeres i en blanding af den sagsorienterede analyserende fagkultur og den dialektisk reflekterende fagkultur? Eller hvad med en fysiklærers behov for både at kunne modellere matematisk, kunne eksperimentere og kunne reflektere over sin undervisning?

Pointen med de fem fagkulturer er altså ikke, at de kan eller skal bruges til at sætte folk i bås. Tværtimod bør de bruges til, at bygge broer over faglige kulturkløfter. For at gøre det er det nødvendigt at kortlægge kløfterne.

Den sædvanlige måde at beskrive universiteternes faggeografi på svarende til administrative opdelinger, røber efter min vurdering ikke, hvor

*kultur*kløfterne er. På f.eks. RUC viser organiseringen i de fire bacheloruddannelser, de seks institutter og de ca. 25 fag umiddelbart mere om administrative kløfter end om kulturkløfter. En fagkombination af det empirisk eksperimenterende kemifag og det matematisk modellerende fysikfag eller et tilsvarende forskningssamarbejde vil normalt – på trods af Trines oplevede kulturforskel mellem kemi og fysik som fag i deres helhed – kunne foregå uden brobygning, da der findes mange veletablerede både/og kulturer i grænseområderne imellem de to fag. Hvorimod f.eks. en kombination af matematik og biologi afgjort forudsætter brobygning, fordi de to fag mere entydigt placerer sig i hver sin fagkultur. En kombination af socialvidenskab og geografi giver heller ikke anledning til store kultursammenstød, hvis det er socialgeografi, som socialvidenskab kombineres med. Hvorimod en kombination af naturgeografi med f.eks. erhvervsøkonomi er et krævende projekt (svarende til Tek-Sam's projekt gennem mange år med at få den empirisk eksperimenterende nat-kultur i miljøet og den sagsorienterede analyserende sam-kultur i miljøet til at spille sammen). RUC's opdeling i institutter og fag er nærmest en opdeling efter emner. Men det er ikke forskelle i emner så meget som forskelle i arbejds- og anskuelsesmåder, der skiller fagkulturerne fra hinanden.

Skitsen af universiteternes fem fagkulturer er altså ment som en beskrivelse af, hvor de store kommunikationsproblemer på tværs af fagene ligger. Eller sagt på en anden måde et forsøg på karakterisering af de forskellige hovedanskuelsesmåder på universiteterne. Eller med figur 1. et forsøg på at beskrive, hvilke slags forskellige polaroidfiltre, der skal erhverves for at kunne manøvrere kosmopolitisk på tværs i universitetslandskabet.

Tværfaglighed og samfundsvidenskab

Ved RUC' s 25 års jubilæumskonference i 1997 om projektpædagogik holdt den daværende repræsentant for de studerende Frederik Voetmann Christiansen et oplæg (Voetmann Christiansen, F, 1999: 57-66), hvor han modstillede realiseringen af projektpædagogikken på den samfundsvidenskabelige basisuddannelse og den naturvidenskabelige basisuddannelse. (Den humanistiske basisuddannelse betragtede han pædagogisk set som en mellemproportional mellem de to. Den humanistisk–teknologiske basisuddannelse eksisterede ikke dengang). På den naturvidenskabelige basisuddannelse er problemorienteringen af projektarbejdet først og fremmest til for at holde de studerendes læreproces på sporet i kraft af, at

problemorienteringen giver dem et sigtepunkt. På den samfundsvidenskabelige basisuddannelse skal problemorienteringen herudover også sikre projektarbejdets samfundsrelevans. Projektarbejdet bliver derfor oftest tværfagligt som en følge af problemorienteringen, fordi "virkelige problemer ikke respekterer faggrænser". Anderledes på den naturvidenskabelige basisuddannelse. Her skal samfundsrelevansen og tværfagligheden sikres ved indretningen af uddannelsen i sin helhed og ikke i det enkelte projekt. Tværtimod lægges der op til, at der også arbejdes med disciplin-definerede problemer i det enkelte projektarbejde for at få fortrolighed med fagenes tænkemåder. Hvorimod det på den samfundsvidenskabelige basisuddannelse er afgørende, at økonomi, sociologi og politologi integreres ved behandlingen af "sagen".

Forskellen i realiseringen af projektpædagogikken på den naturvidenskabelige og den samfundsvidenskabelige basisuddannelse hænger ifølge Frederik Voetmann Christiansen sammen med, hvor fasttømrede paradigmer de pågældende fag har. Med min tilspidsning: På den samfundsvidenskabelige basisuddannelse er tværfagligheden nødvendig, fordi fagene er under opløsning. De belyser ikke "sagerne" tilstrækkeligt hver for sig. På den naturvidenskabelige basisuddannelse er tværfagligheden nødvendig, fordi fagene *ikke* er under opløsning. Derfor er der behov for brobygning og tolkning imellem dem. Den måde tværfaglighedsdiskussionen typisk føres på, afspejler således fagkulturen, hvori diskussionen foregår.

Den fremherskende opfattelse af ordet tværfaglighed inden for samfundsvidenskab og den sagsorienterede analyserende fagkultur svarer i det hele taget til nummer 2 (fagintegration bestemt af et praktisk problem- og funktionsfelt) af mine tre skitserede forståelser af tværfaglighed (se side XX). Tværfaglighed forstås som en konsekvens af problemorienteringen i forhold til en sag. Som naturvidenskabsdidaktiker er jeg indforstået med denne forståelse. Jeg regner naturvidenskabsdidaktik for at høre til den sagsorienterede analyserende fagkultur. Men i øvrigt har det jo ikke været denne forståelse af tværfaglighed mit bidrag til bogen her har skullet belyse. Det jeg har forsøgt at illustrere og understrege med mit bidrag er, at ordet tværfaglighed også bruges i forbindelse med ambitioner om, at der i uddannelser og forskning bygges broer på tværs af fagkløfterne i universitetslandskabet. Altså svarende til den tredje betydning i tværfaglighedssskemaet, hvor der sigtes til forståelse og overblik på tværs af fag. Jeg har også villet illustrere, at universitetslandskabet, der skal bygges broer i, består af mere end den sagsorienterede analyserende fagkultur.

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Artiklen side 4 i European Journal of Physics gav anledning til en replik i European Journal of Physics. På denne baggrund sendte jeg en justeret artikel om rulning med nye eksempler, men med de samme hovedbudskaber, som den første, til American Journal of Physics. Denne gang blev den optaget.

Five ways of deriving the equation of motion for rolling bodies

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The equation of motion for rolling bodies can be derived in several different ways. This paper shows that the equation of motion can be derived in five and only five ways without needing to know the normal and frictional forces acting at the point of contact. After enumerating these five ways, we examine two illustrative examples: an asymmetric disk rolling on a cylinder and a symmetric ball rolling on a turntable. We also discuss the educational benefits of including this topic when teaching mechanics. © *2012 American Association of Physics Teachers.*
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I. INTRODUCTION

In a recent paper, Turner and Turner discussed the dynamics of a rolling asymmetric body on a horizontal plane.[1] By providing a counterexample, they disproved the widespread misunderstanding that the torque equation, $d\mathbf{L}/dt = \boldsymbol{\tau}$, must always be applied about the instantaneous point of contact, see Ref. 2 for examples of the misunderstanding. Turner and Turner derived the correct equation of motion by modifying the torque equation with an additional term that they refer to as a phantom torque.

In a recent paper I discussed two equations to be used instead of $\boldsymbol{\tau} = d\mathbf{L}/dt$ in order to derive the equation of motion of rolling bodies by taking moments about the point of contact.[2] Calling the point of contact Q, the angular momentum about Q, $\mathbf{L}_Q$, apparently can be understood in two different ways leading to two different replacements of $d\mathbf{L}_Q/dt = \boldsymbol{\tau}_Q$.

If $\mathbf{L}_Q$ is understood as $\mathbf{L}_{QI} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \mathbf{v}_i$, the correct equation is

$$\frac{d\mathbf{L}_{QI}}{dt} = \boldsymbol{\tau}_Q + M\mathbf{v}_{CM} \times \mathbf{v}_Q. \tag{1}$$

Here, $\mathbf{r}_i$ are the position vectors in an inertial frame of reference of the mass points m_i of the rolling body, $\mathbf{r}_Q$ is the position vector of Q in the inertial frame of reference, $\mathbf{v}_i = d\mathbf{r}_i/dt$, $\mathbf{v}_Q = d\mathbf{r}_Q/dt$, M is the total mass of the body, and $\mathbf{v}_{CM}$ is the velocity of the center of mass of the body in the chosen inertial frame of reference. The index Q in $\mathbf{L}_{QI}$ refers to the point about which moments are taken. The index I indicates that the velocities in the definition of $\mathbf{L}_{QI}$ are evaluated in an inertial frame of reference.

If, on the other hand, $\mathbf{L}_Q$ is understood as $\mathbf{L}_{QQ} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i(\mathbf{v}_i - \mathbf{v}_Q)$, the velocities thus evaluated relative to Q, the correct equation is

$$\frac{d\mathbf{L}_{QQ}}{dt} = \boldsymbol{\tau}_Q + (\mathbf{r}_{CM} - \mathbf{r}_Q) \times \left(\frac{-Md\mathbf{v}_Q}{dt}\right). \tag{2}$$

Here, as in Eq. (1), $\boldsymbol{\tau}_Q$ is the torque due to the forces existing in the inertial frame of reference. The first index Q in $\mathbf{L}_{QQ}$ again refers to the point about which moments are taken. The second index Q indicates that velocities in the definition of $\mathbf{L}_{QQ}$ are evaluated relative to Q.

In Ref. 2, it was also clarified that the contact point, Q, can be defined either as the point attached to the rolling body instantly at the point of contact or as the geometrical point defined as the point of contact at any time. What is meant by both $\mathbf{v}_Q$ and $d\mathbf{v}_Q/dt$ in Eqs. (1) and (2) depends of course on what is meant by Q. Altogether this gives four different ways of deriving the equation of motion for a rolling body without needing to know the normal and frictional forces acting at Q. Turner and Turner used Eq. (2) with Q understood as a point attached to the rolling body.

Very recently Hu has pointed out that the equation of motion of a rolling body can also be derived without needing to know the forces acting at Q by defining Q as the point attached to the ground instantly at the point of contact.[3] Differentiating $\mathbf{L}_Q$ written as a sum of its orbital part, $\mathbf{L}_{\text{orb}} = (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_{CM}$, and its spin part, $\mathbf{L}_{\text{spin}} = \mathbf{L}_{CM}$, then gives

$$\boldsymbol{\tau}_Q = (\mathbf{r}_{CM} - \mathbf{r}_Q) \times \frac{Md\mathbf{v}_{CM}}{dt} + \frac{d\mathbf{L}_{CM}}{dt}. \tag{3}$$

Thus, we have five different ways of deriving the equation of motion for a rolling body without needing to know the forces at the point of contact. The five different ways are summarized in Table I for cases with the angular momentum parallel to the angular velocity $\boldsymbol{\omega}$, as is the case, e.g., for rolling motion in two dimensions where $\mathbf{L}$ and $\boldsymbol{\omega}$ are both perpendicular to the plane of the rolling motion, independent of whether $\mathbf{L}$ is $\mathbf{L}_{QI}$, $\mathbf{L}_{QQ}$, or $\mathbf{L}_{CM}$. In Table I, $\mathbf{v}_{Qi}$ and $d\mathbf{v}_Q/dt|_i$ are the instantaneous values of $\mathbf{v}_Q$ and $d\mathbf{v}_Q/dt$, and I_Q is the moment of inertia relative to Q. For details, see Ref. 2.

Contrary to the situations where Q is defined as the point of contact at any time or a point attached to the rolling body, it makes no difference evaluating velocities relative to the laboratory or relative to Q, when Q is a point attached to the ground. Thus, we have five, not six, different ways. Since the ambiguous term "the instantaneous point of contact," Q, cannot be understood in more ways than the three mentioned and it is hard to give $\mathbf{L}_Q$ other meanings than $\mathbf{L}_{QI}$ and $\mathbf{L}_{QQ}$, there are five and only five ways of deriving the equation of motion for bodies without needing to know the forces acting at the point of contact.

Turner and Turner analyzed the motion of a semicircular hoop rocking on a horizontal plane using method **D**.[1] I demonstrated the four methods **A** through **D** used on the same case,[2] whereas Hu in his demonstration of method **E** considered the more general case of asymmetric disks rolling on an inclined plane, including the rocking hoop as a special case.[3] In Sec. II, the further generalized case of asymmetric disks rolling on cylinders is considered. The purpose is to demonstrate the value of having access to all five different ways of

Table I. Overview of the five ways of deriving the equation of motion for rolling bodies without needing to know the forces acting at the point of contact.

	Choose	Use	Note
A.	$\mathbf{L}_Q = \mathbf{L}_{QI}$; Q the point of contact at any time	Eq. (1)	$\mathbf{v}_{Qi} \neq 0$; $\mathbf{L}_{QI} = I_Q\boldsymbol{\omega}$; $\frac{dI_Q}{dt} \neq 0$ in general
B.	$\mathbf{L}_Q = \mathbf{L}_{QI}$; Q attached to the rolling body	Eq. (1)	$\mathbf{v}_{Qi} = 0$; $\mathbf{L}_{QI} = I_Q\boldsymbol{\omega} + (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_Q$; $\frac{dI_Q}{dt} = 0$
C.	$\mathbf{L}_Q = \mathbf{L}_{QQ}$; Q the point of contact at any time	Eq. (2)	$\mathbf{v}_{Qi} \neq 0$; $\mathbf{L}_{QQ} = I_Q\boldsymbol{\omega} - (\mathbf{r}_{CM} - \mathbf{r}_Q) \times M\mathbf{v}_Q$; $\frac{dI_Q}{dt} \neq 0$
D.	$\mathbf{L}_Q = \mathbf{L}_{QQ}$; Q attached to the rolling body	Eq. (2)	$\mathbf{v}_{Qi} = 0$; $\mathbf{L}_{QQ} = I_Q\boldsymbol{\omega}$; $\frac{dI_Q}{dt} = 0$; $\frac{d\mathbf{v}_Q}{dt}\vert_i \neq 0$
E.	Q attached to the ground	Eq. (3)	$\mathbf{v}_Q = 0$; $\mathbf{L}_{QI} = \mathbf{L}_{QQ}$

deriving the equation of motion in order to be able to choose the easiest one for the case in consideration. In Sec. III, a symmetric ball rolling on a turntable is discussed as a further illustration. The two cases of Secs. II and III motivate Sec. IV, where it is argued that method **A** is usually the easiest of the five ways. Section IV argues further that the seldom noticed,[2,4,5] Eq. (1) gives the easiest insight into whether the torque equation applied about the instantaneous point of contact may be used in its simple form $d\mathbf{L}_{QI}/dt = \boldsymbol{\tau}_Q$ or whether it must be supplemented with extra terms. As mentioned, the widespread justification of using the torque rule in its simple form is based on a misunderstanding. Section V discusses potential educational benefits of pointing out the misunderstanding, in university courses in mechanics, and working with its possible replacements.

The calculations in Secs. II–IV are presented in rather condensed form. The reader has to either accept the calculation details and focus on the results or take paper and pencil in order to check the calculations. The calculations are presented in condensed form in order not to overshadow the main message of the article. The main message is that working with the five different ways of deriving the equation of motion for rolling bodies illuminates how arguments in physics can be critically tested by going back to basic definitions and assumptions.

II. ROLLING OF ASYMMETRIC DISKS ON CYLINDERS

We proceed to derive the equations of motion for the asymmetric disks rolling on the cylinders shown in Fig. 1. The direction of gravity is downwards. First, we focus on the case of Fig. 1(a). If $\mathbf{x}_u$, $\mathbf{y}_u$, $\mathbf{z}_u$ are defined as unit vectors in the laboratory frame of reference as indicated in the figure, the unit vectors **a** and **c** (also indicated in the figure) are given as $\mathbf{a} = \sin(\theta+\phi)\,\mathbf{x}_u - \cos(\theta+\phi)\,\mathbf{y}_u$ and $\mathbf{c} = \sin\phi\,\mathbf{x}_u - \cos\phi\,\mathbf{y}_u$. In each of the five possible derivations, $\boldsymbol{\tau}_Q$ is consequently given by

$$\begin{aligned}\boldsymbol{\tau}_Q &= \mathbf{r}_{QCM} \times Mg(-\mathbf{y}_u) = (-r\mathbf{c} + \beta r\mathbf{a}) \times Mg(-\mathbf{y}_u)\\ &= Mgr(\sin\phi - \beta\sin(\theta+\phi))\mathbf{z}_u\,. \end{aligned} \tag{4}$$

Here, $\mathbf{r}_{QCM}$ is the vector from Q to CM (the center of mass of the disk), M is the mass of the disk, g the strength of the gravitational field, r the radius of the disk, βr is the distance of the center of mass from the geometrical center of the disk, and θ and ϕ are the angles indicated in the figure. Also the rolling condition (R being the radius of the cylinder),

$$R\frac{d\phi}{dt} = -r\frac{d\theta}{dt}, \tag{5}$$

applies for all five ways of deriving the equation of motion.

In order to continue the derivation from here one has to decide which of the five ways to follow. Here, we follow method **A**, i.e., we use Eq. (1) and identify Q as the geometrical point defined as the point of contact at any time.

Since $\mathbf{r}_{QCM}^2 = (-r\mathbf{c} + \beta r\mathbf{a})^2 = r^2(1+\beta^2 - 2\beta\mathbf{c}\cdot\mathbf{a}) = r^2(1+\beta^2 - 2\beta\cos\theta)$, we now have $dI_Q/dt = d(I_{CM} + M\mathbf{r}_{QCM}^2)/dt = 2Mr^2\beta\sin\theta\cdot d\theta/dt$. Inserting this into

$$\begin{aligned}\frac{d\mathbf{L}_{QI}}{dt} &= \frac{d\left(I_Q\left(\frac{d\theta}{dt}+\frac{d\phi}{dt}\right)\mathbf{z}_u\right)}{dt} = \frac{d\left(I_Q\left(1-\frac{r}{R}\right)\frac{d\theta}{dt}\mathbf{z}_u\right)}{dt}\\ &= \left(\frac{dI_Q}{dt}\cdot\left(1-\frac{r}{R}\right)\frac{d\theta}{dt} + I_Q\cdot\left(1-\frac{r}{R}\right)\frac{d^2\theta}{dt^2}\right)\mathbf{z}_u, \end{aligned} \tag{6}$$

where Eq. (5) was used, we get

$$\begin{aligned}\frac{d\mathbf{L}_{QI}}{dt} = \Bigg(&\left(1-\frac{r}{R}\right)\cdot I_Q\cdot\frac{d^2\theta}{dt^2}\\ &+2\left(1-\frac{r}{R}\right)\cdot\beta Mr^2\sin\theta\cdot\left(\frac{d\theta}{dt}\right)^2\Bigg)\mathbf{z}_u \end{aligned} \tag{7}$$

for the left side of Eq. (1). The third term in Eq. (1) is derived from

$$\begin{aligned}\mathbf{v}_{CM} &= \frac{d((R-r)\mathbf{c} + \beta r\mathbf{a})}{dt}\\ &= (R-r)\mathbf{d}\cdot\frac{d\phi}{dt} + \beta r\mathbf{b}\cdot\frac{d(\theta+\phi)}{dt}\\ &= \left(1-\frac{r}{R}\right)\mathbf{d}\cdot\frac{-rd\theta}{dt} + \beta r\mathbf{b}\cdot\left(1-\frac{r}{R}\right)\frac{d\theta}{dt}, \end{aligned} \tag{8}$$

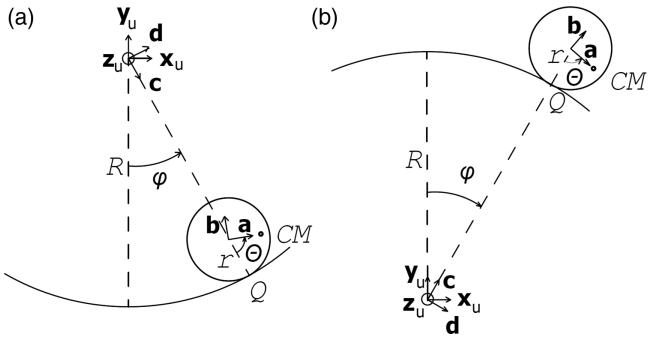

Fig. 1. (a) Asymmetric disc rolling inside a cylinder. (b) Asymmetric disc rolling on top of a cylinder. In both figures, $\mathbf{x}_u$, $\mathbf{y}_u$, and $\mathbf{z}_u$ are unit vectors in the laboratory system. Also, in both figures **a** is a unit vector directed from the centers of the disks towards the centers of mass (*CM*) of the disks, **b** a unit vector perpendicular to **a**, **c** a unit vector directed from the centers of the cylinders to the centers of the disks, and **d** a unit vector perpendicular to **c**. The direction of gravity is downwards.

and

$$\mathbf{v}_Q = \frac{d(R\mathbf{c})}{dt} = R\mathbf{d}\cdot\frac{d\phi}{dt} = -r\mathbf{d}\cdot\frac{d\theta}{dt}\,, \tag{9}$$

where we have used Eq. (5) and **b** and **d** are unit vectors perpendicular to **a** and **c** as shown in Fig. 1(a). Since $\mathbf{b}\times\mathbf{d} = -\sin\theta\cdot\mathbf{z}_u$, we get

$$M\mathbf{v}_{CM}\times\mathbf{v}_Q = \left(1-\frac{r}{R}\right)\cdot\beta Mr^2\sin\theta\cdot\left(\frac{d\theta}{dt}\right)^2\cdot\mathbf{z}_u. \tag{10}$$

Inserting the results of Eqs. (4), (7), and (10) into Eq. (1), we conclude that the equation of motion for an asymmetric disk rolling inside a cylinder is

$$\left(1-\frac{r}{R}\right)\cdot I_Q\cdot\frac{d^2\theta}{dt^2} + \left(1-\frac{r}{R}\right)\cdot\beta Mr^2\sin\theta\cdot\left(\frac{d\theta}{dt}\right)^2 = Mgr(\sin\phi - \beta\sin(\theta+\phi)). \tag{11}$$

From this equation and the proper initial conditions, θ can be found as a function of time since $\phi = (r/R)(\theta_0-\theta)$ according to the rolling condition in Eq. (5), where θ_0 is the value of θ when $\phi = 0$. One can also use $\theta = \theta_0 - (R/r)\phi$ to transform Eq. (11) to a differential equation involving ϕ as a function of time.

With the sign conventions of Fig. 1(b), by similar calculations, the equation of motion for an asymmetric disk rolling on top of a cylinder corresponding to Eq. (11) is found to be

$$\left(1+\frac{r}{R}\right)\cdot I_Q\cdot\frac{d^2\theta}{dt^2} + \left(1+\frac{r}{R}\right)\cdot\beta Mr^2\sin\theta\cdot\left(\frac{d\theta}{dt}\right)^2 = Mgr\Big(\sin(-\phi) - \beta\sin(\theta-\phi)\Big). \tag{12}$$

For $R \gg r$ where ϕ can be regarded as constant in time compared to θ and $(1-r/R)\approx(1+r/R)\approx 1$, both Eqs. (11) and (12) reproduce the result found by Hu[3] for an asymmetric disk rolling on an inclined plane as a limiting case.

III. SYMMETRIC BALL ROLLING ON A TURNTABLE

In order to discuss which of the five ways is the easiest it is also instructive to look at the case of a symmetric ball rolling on a turntable, as shown in Fig. 2.

We identify Q as the point of contact at any time. Denoting the radius of the ball by r, we have $\mathbf{r}_{CM} = \mathbf{r}_Q + r\cdot\mathbf{e}$, where **e** is a unit vector as shown in Fig. 2. Thus, the velocities of *CM* and Q relative to the laboratory $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are equal. Since $\boldsymbol{\tau}_Q = 0$, assuming the turntable to be in a horizontal plane and $\mathbf{v}_{CM}\times\mathbf{v}_Q = 0$, Eq. (1) of method **A** simply gives $d\mathbf{L}_{QI}/dt = 0$. Denoting the mass of the ball by M and its moment of inertia around its center of mass by kMr^2, we have $\mathbf{L}_{QI} = r\mathbf{e}\times M\mathbf{v}_{CM} + kMr^2\boldsymbol{\omega}$ where $\boldsymbol{\omega}$ is the angular velocity of the ball relative to the laboratory. (The separation $\mathbf{L}_{QI} = \sum_i m_i(\mathbf{r}_i-\mathbf{r}_Q)\times\mathbf{v}_i = \sum_i m_i(\mathbf{r}_i-\mathbf{r}_{CM})\times\mathbf{v}_i + \sum_i m_i(\mathbf{r}_{CM}-\mathbf{r}_Q)\times\mathbf{v}_i = kMr^2\cdot\boldsymbol{\omega} + r\cdot\mathbf{e}\times M\mathbf{v}_{CM}$ of $\mathbf{L}_{QI}$ into a spin and an orbit part is valid for any chosen Q.) Inserting this equation into $d\mathbf{L}_{QI}/dt = 0$, we get

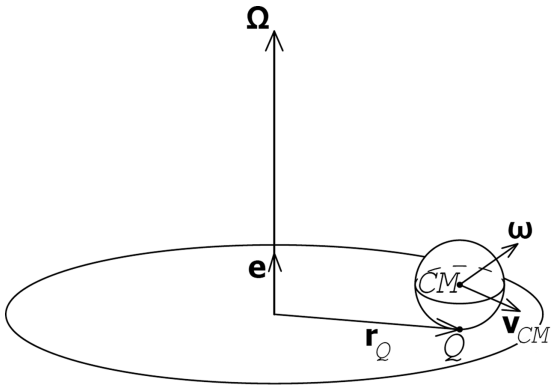

Fig. 2. Symmetric ball rolling on a turntable. Here, Ω is the angular velocity of the turntable, **e** is a unit vector in the direction of Ω, $\mathbf{r}_Q$ is the position vector of the point of contact Q, $\boldsymbol{\omega}$ is the angular velocity of the ball relative to the laboratory, and $\mathbf{v}_{CM}$ is the velocity of the center of mass (*CM*) of the ball relative to the laboratory.

$$r\cdot\frac{d\boldsymbol{\omega}}{dt} = -\frac{1}{k}\cdot\mathbf{e}\times\frac{d\mathbf{v}_{CM}}{dt}. \tag{13}$$

Note that this equation is nothing but a consequence of the Newtonian laws of motion (ultimately it is a consequence of Newton's second and third laws for particles since the laws for the motion of a system of particles are derived from these). In order to find the equation of motion for the ball on the turntable, besides the Newtonian constraint between $\boldsymbol{\omega}(t)$ and $\mathbf{v}_{CM}(t)$ of Eq. (13), we must add the rolling condition constraint between $\boldsymbol{\omega}(t)$ and $\mathbf{v}_{CM}(t)$. The velocity relative to the laboratory of the bottom point of the ball is $\mathbf{v}_{CM} + \boldsymbol{\omega}\times(-r\mathbf{e})$. The velocity relative to the laboratory of the point of the turntable touching the ball is $\Omega\times\mathbf{r}_Q$, where Ω is the angular velocity of the turntable. The condition for rolling without sliding is therefore

$$\Omega\times\mathbf{r}_Q = \mathbf{v}_{CM} + \omega\times(-r\mathbf{e}). \tag{14}$$

Differentiating Eq. (14) with respect to time and inserting Eq. (13), we get $\Omega\times\mathbf{v}_Q = d\mathbf{v}_{CM}/dt - (1/k)\cdot(\mathbf{e}\times d\mathbf{v}_{CM}/dt)\times(-\mathbf{e})$ and

$$\frac{d\mathbf{v}_{CM}}{dt} = \frac{k}{1+k}\cdot(\Omega\times\mathbf{v}_Q), \tag{15}$$

because $(\mathbf{e}\times d\mathbf{v}_{CM}/dt)\times\mathbf{e} = d\mathbf{v}_{CM}/dt$. Since Q was chosen to be the point of contact at any time, $\mathbf{v}_Q$ and $\mathbf{v}_{CM}$ are equal. Therefore, Eq. (15) shows that the acceleration of the *CM* at any time is perpendicular to the velocity of the *CM* and thus, of constant magnitude. Seen from the laboratory, the ball is simply performing a uniform circular motion! Equation (15) shows that the angular frequency in the circular motion is $k/(1+k)\cdot\Omega$. The center and the radius of the circle are determined by the initial conditions.

Let us denote the unknown static friction force between ball and turntable by **G**. The torque rule about the center of mass gives $kMr^2\cdot d\boldsymbol{\omega}/dt = (-r\mathbf{e})\times\mathbf{G}$. If we combine this with Newton's second law for the motion of the center of mass, $Md\mathbf{v}_{CM} = \mathbf{G}$, Eq. (13) is arrived at. Consequently, the advantage of taking the torque about the point of contact Q and thus not needing to introduce **G** is limited in the case of a symmetric ball rolling on a turntable. My purpose of discussing this case is not to advocate for one of the five ways

of taking moments about Q, but to discuss which way is the easiest among the five.

IV. THE EASIEST WAY OF DERIVING THE EQUATION OF MOTION

Which of the five ways is the easiest? Let us consider the two cases in Secs. II and III, starting with the rolling of asymmetric disks on cylinders.

As mentioned, the equations of motion (11) and (12) can be derived without needing to know the forces acting at the point of contact in four ways different from what we did above. However, the four other ways are not as easy as method **A**. Method **E** used by Hu[3] is more troublesome because calculating $d\mathbf{v}_{CM}/dt$ involves differentiation of the rotating unit vectors to second order instead of only to first order when calculating $\mathbf{v}_{CM}\times\mathbf{v}_Q$ using method **A**. According to method **D** used by Turner and Turner,[1] instead of $d\mathbf{v}_{CM}/dt$ we have to derive $d\mathbf{v}_Q/dt$. This also involves differentiation of the rotating unit vectors to second order and presents difficulties similar to those using method **E**, since Q for method **D** is understood as a point attached to the rolling body as the CM is. Method **B** leads to the same calculations as method **D**. Differentiating $\mathbf{L}_{QI}=I_Q\boldsymbol{\omega}+(\mathbf{r}_{CM}-\mathbf{r}_Q)\times M\mathbf{v}_Q$ with respect to time and inserting the result into Eq. (1) results in the same equation as a starting point for further calculations as inserting $\mathbf{L}_{QQ}=I_Q\boldsymbol{\omega}$ into Eq. (2). Finally, method **C** leads to the same calculations as method **A**. Differentiating $\mathbf{L}_{QQ}=I_Q\boldsymbol{\omega}-(\mathbf{r}_{CM}-\mathbf{r}_Q)\times M\mathbf{v}_Q$ with respect to time and inserting the result into Eq. (2) gives the same equation as a starting point for further calculations as inserting $\mathbf{L}_{QI}=I_Q\boldsymbol{\omega}$ into Eq. (1).

The five different ways of deriving the equation of motion for rolling bodies without needing to know the normal and frictional forces acting at the point of contact thus reduce to three different sets of calculations. In the case of rolling of asymmetric disks on cylinders the three sets of calculations differ quite a bit. Using method **E** and method **D** (or method **B** including the same calculations) involves differentiation of the rotating unit vectors to second order, whereas method **A** (or method **C** including the same calculations) involves only differentiation to first order. As method **C** is conceptually more awkward than method **A**, this makes method **A** the easiest way of the five for the case of rolling of asymmetric disks on cylinders.

We now turn to the symmetric ball rolling on a turntable. Besides following method **A** as in Sec. III, it is also possible to use method **C**. Here Q is also understood as the point of contact at any time and therefore $\mathbf{r}_{CM}=\mathbf{r}_Q+r\mathbf{e}$ and $\mathbf{v}_{CM}=\mathbf{v}_Q$. Consequently, we have $\boldsymbol{L}_{QQ}=kMr^2\boldsymbol{\omega}$ (when velocities are evaluated relative to Q and Q follows the CM, the orbital part of the angular momentum is zero). Inserting this into Eq. (2) with $\boldsymbol{\tau}_Q=0$ and $\mathbf{v}_Q=\mathbf{v}_{CM}$ we get again Eq. (13). But if Q is understood as a point attached to the ball or as a point attached to the turntable it is only instantaneously that $\mathbf{r}_{CM}$ equals $\mathbf{r}_Q+r\mathbf{e}$, and $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are therefore generally not identical. For this reason, in the case of a symmetric ball rolling on a turntable, methods **B**, **D**, and **E** are difficult to apply.

We could also ask which is the easiest way of justifying the use of $d\mathbf{L}_Q/dt=\boldsymbol{\tau}_Q$ for the case of rolling of symmetric disks on an inclined plane. This is the special case of rolling of asymmetric disks on cylinders defined by $\beta=0$ and $r/R=0$ and the textbook case where the application of the normal torque rule about the instantaneous point of contact (Q) has often been justified by the incorrect argument that Q can be considered a fixed point since $\mathbf{v}_{Qi}=0$. From Eq. (11), putting $\beta=0$ and $r/R=0$ it is seen that $d\mathbf{L}_Q/dt=\boldsymbol{\tau}_Q$ leads to the correct equation of motion,

$$I_Q\cdot\frac{d^2\theta}{dt^2}=Mgr\cdot\sin\theta. \tag{16}$$

Using the simple torque rule leads to the correct equation in this case. But how do we give a correct justification for doing it? Since $\mathbf{L}_Q$ can be understood in two ways and Q in three ways, each of the five ways mentioned may offer the justification. Let us restrict ourselves to the three methods **A**, **D**, and **E**. Methods **B** and **C** are more complicated and do not bring new perspectives. Why does the simple torque rule give the correct result? According to method **A** and Eq. (1), it is because $\mathbf{v}_{CM}\times\mathbf{v}_Q$ in this case is zero since $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are parallel, Q being the geometrically defined point of contact. According to method **D** and Eq. (2) it is because $(\mathbf{r}_{CM}-\mathbf{r}_Q)\times(-Md\mathbf{v}_Q/dt)$ is zero since $\mathbf{r}_{CM}-\mathbf{r}_Q$ and $Md\mathbf{v}_Q/dt|_i$ are parallel, Q being now attached to the rolling body. According to method **E** it is because $|(\mathbf{r}_{CM}-\mathbf{r}_Q)\times Md\mathbf{v}_{CM}/dt|=Mr^2\cdot d^2\theta/dt^2$, using the rolling condition $v_{CM}=r\cdot d\theta/dt$, and $|d\mathbf{L}_{CM}/dt|=I_{CM}\cdot d^2\theta/dt^2$, which combined with $I_Q=Mr^2+I_{CM}$ gives Eq. (16) from Eq. (3). Clearly the three ways are quite different, with **A** being, perhaps, the easiest of them.

The three cases investigated thus indicate method **A** to be the easiest way. It is interesting that method **A** involves the seldom noticed Eq. (1). More generally, this equation also gives the easiest insight into whether the torque rule applied about the instantaneous point of contact may be used in its simple form $d\mathbf{L}_{QI}/dt=\boldsymbol{\tau}_Q$ or whether extra terms must be added. Deviations from the simple torque rule arise from one of two causes. Either $\mathbf{v}_{CM}$ and $\mathbf{v}_Q$ are not parallel and the term $M\mathbf{v}_{CM}\times\mathbf{v}_Q$ in Eq. (1) is consequently not zero because the rolling body is asymmetric, or $M\mathbf{v}_{CM}\times\mathbf{v}_Q$ is nonzero because the body rolls on a curved surface giving rise to a component of $\mathbf{v}_{CM}$ perpendicular to $\mathbf{v}_Q$. This applies even for symmetric bodies.

Awareness of the different possibilities of deriving the equation of motion for rolling bodies without needing to know the forces acting at the point of contact could perhaps in some cases be of greater importance than in the cases treated here, e.g., when dealing with a more complicated rolling in three dimensions. It is therefore important to note that Eqs. (1)–(3) apply for rolling in three dimensions, as well as for rolling in two dimensions.

V. POTENTIAL EDUCATIONAL YIELDS: ILLUSTRATING PHYSICS AS A RATIONALISTIC SCIENCE

The main message of the article is not the calculations themselves but the potential educational yields of dealing with the calculations and their results. Working with rolling problems and investigating the relative applicability of the five methods will obviously give students extensive mathematical training. But besides this it also gives the students insights into the nature of physics as a scientific discipline.

In my own teaching at the third year at Roskilde University, I show the students how to derive Eq. (15) and thus predict that the ball on the turntable performs uniform circular

motion and does not fly off the edge. I then emphasize to the students that this surprising result is ultimately deduced from nothing but Newton's second law for the motion of point particles and Newton's third law for the action and reaction between two point particles (assuming that the static friction coefficient is high enough to insure rolling without sliding). Thus, going from Newton's second and third laws for point particles all the way to the uniform circular motion on the turntable is merely a matter of mathematical deductions.

Physics is often characterized as an empirical science. The case of rolling bodies demonstrates that physics is also a rationalistic science. In common with chemistry and biology, physics has an empirical aspect. Thereby, physics leans on the philosophical doctrine of empiricism (that all knowledge of matters of facts derives from experience and mind is not furnished with a set of concepts in advance of experience). But physics also makes common cause with mathematics by being a rationalistic science, leaning on the philosophical doctrine of rationalism (that the exercise of reason, rather than experience, provides the primary basis for knowledge). These two aspects of physics, the empirical and the rationalistic, are equally important and must both be included to fully characterize the scientific nature of physics.

Not only physics but also physicists like myself may act extremely rationalistically. When working on writing up the motion of the ball rolling on the turntable, I first made the calculations in the rotating frame of reference. These were troublesome calculations involving the integration of the different torque contributions from the centrifugal and Coriolis forces acting on the different parts of the ball. Eventually, I found the track on the turntable to be a cycloid. From this I deduced the surprising circular motion seen from the laboratory frame of reference. Then—and only then—did I place a ping-pong ball on my rotating gramophone disc and confirm the prediction. Assisted by a colleague I then performed the much easier calculations in the laboratory system. I also found that my "discovery" was in fact a rediscovery of a rediscovery. The circular orbits had been reported by Weltner[6] in 1979 followed by a note by Romer[7] in 1981 pointing out that the phenomenon dates back in the literature to at least 1844. In the note Romer also mentioned that even for him it was only late in his work with the phenomenon that he tested the theoretical prediction of circular orbits experimentally.

The point of the story about the ball on the turntable is the emphasis on the possibility of demonstrating the rationalistic side of physics when teaching classical mechanics. The subject of this article is also suitable for this purpose. For undergraduate students, a presentation of the five derivation methods described here can be used as an example to demonstrate that textbooks—even in physics—are not always correct. It can also be used to demonstrate that in a subject like classical mechanics, mistakes can be straightened out by critical analysis, building on the basic definitions and assumptions. For graduate students in physics, rolling bodies give an example of experiencing physics as a subject where arguments can be critically tested. By working with rolling problems of all kinds, investigating the internal consistency among the five methods, students by themselves can experience physics as a rationalistic science.

VI. SUMMARY

The equation of motion for rolling bodies can be derived in five different ways without needing to know the normal and frictional forces acting at the point of contact. In general, it is not correct to just equate the rate of change of the angular momentum about the point of contact with the resulting moments about the point of contact of the forces operating in the laboratory frame of reference. The five ways differ because both the term "the point of contact" and the term "angular momentum" are ambiguous. The point of contact Q can be understood as a point attached to the rolling body, as a point attached to the ground, or as the geometrically defined point of contact at any time. Moreover, $\mathbf{L}_Q$ can be understood as both $\mathbf{L}_{QI} = \sum_i (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \mathbf{v_i}$ and $\mathbf{L}_{QQ} = \sum_i (\mathbf{r}_i) - \mathbf{r}_Q \times m_i (\mathbf{v}_i - \mathbf{v}_Q)$.

The easiest way of the five ways is usually to choose Q as the geometrically defined point of contact at any time and identify $\mathbf{L}_{QI}$ with $\mathbf{L}_Q$. This implies the use of the seldom-noticed Eq. (1).

The five different ways of deriving the equation of motion for rolling bodies can be used as a case for graduate students to work with for themselves and for undergraduate students to be told about, in both cases with the purpose of illuminating how arguments in physics can be critically tested by going back to the basic definitions and assumptions.

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Det følgende indlæg blev bragt i RUCnyt-4-2012/2013, RUCs daværende meddelelses- og debatblad. Det var en reaktion på den varslede skamfering af RUCs kombinationsuddannelsesstruktur, som indtil da havde været en vigtig del af RUCs profilering i de foregående 20 år.

Debat

"Design din egen uddannelse"

Af Jens Højgaard Jensen, lektor i fysik, tidligere prorektor for RUC og tidligere studieleder for den naturvidenskabelige basisuddannelse.

Hvorfor studere på RUC? Hvad skal svaret være ved cafebordet på Vesterbro student og student imellem? Hvorfor foretrække RUC frem for Københavns Universitet, CBS eller nu Aalborg Universitet, København? De ligger i modsætning til RUC alle inden for cykelafstand.

Svaret må nødvendigvis handle om, hvad RUC kan tilbyde, som de andre ikke kan. Det kan ikke handle alene om klassisk grundfaglighed. RUC vil så fremstå som andetvalget efter Københavns Universitet. Men det kan heller ikke handle alene om erhvervsrettet og sagsorienteret anvendt faglighed. Her vil RUC være andetvalget efter CBS og AAU København. Og problemorienteret projektarbejde markedsfører AAU sig jo på som deres særkende, "Aalborg modellen", i Storkøbenhavn. Set fra RUC er det sidste selvfølgelig hård kost al den stund Aalborg modellen (fra 1974) er et kopiprodukt med RUC modellen (fra 1972) som originalen. Men for potentielt kommende studerende på RUC betyder den forhistorie nok meget lidt. Så, hvorfor tage togturen til RUC?

Det skal man, hvis man er interesseret i *selv at designe sin egen uddannelse.*

På RUC har vi afskaffet fakultetsopdelingen i det klassiske universitet samtidigt med, at vi stadig har opretholdt en bred fagvifte spændende over humaniora, samfundsvidenskab og naturvidenskab. Ophævelsen af fakultetsopdelingen har gjort det muligt at skabe en fælles uddannelsesstruktur på tværs af RUC. De fire brede bachelorindgange med valgmuligheder og indbygget gradvist studievalg inviterer hver enkelt studerende til at udvikle sit helt eget studiedesign. Oven på det sætter kombistrukturen trumf på med det frie valg af to fag på tværs af hele universitetsfagspektret. RUC's helt særlige studiestruktur giver i langt højere grad end de andre universiteter de studerende muligheder for at designe deres egen uddannelse.

Tilbuddet om at kunne kombinere inden for og på tværs af hum, sam og nat og på tværs af grundfag og anvendte fag i en åben struktur med brede indgange er noget særligt for RUC. Samtidigt imødekommer uddannelsen af faglige brobyggere et samfundsmæssigt behov, hvor de andre universiteter i Storkøbenhavn, på grund af måderne de er organiseret på, i mange år vil være forhindret i at tilbyde noget tilsvarende. På RUC kan man noget, der ikke er muligt andre steder: Designe sin egen uddannelse.

Det er ikke nødvendigt at udvikle nye uddannelser på RUC. Vores fagvifte dækker bredt og er i forvejen stor i forhold til vores størrelse. Nye uddannelser kan måske sikre ekstra rekruttering til nydannelserne isoleret set, men øger næppe rekrutteringen til RUC som helhed på sigt. Der er brug for, at RUC markedsføres som en helhed og ikke stykkevis og delt. "Design din egen uddannelse" er en god overskrift for markedsføring af RUC som en helhed.

Men hvis RUC skal være kendt som stedet, hvor der er særlige muligheder for at designe sin egen uddannelse, skal der mere til end den nuværende markedsføring. For at få budskabet frem til cafebordet på Vesterbro skal markedsføringen være fokuseret på budskabet og så massiv, at den sætter dagsorden for en stund. Det kunne være i starten af 2013.

Alternativet for RUC til at være universitetet i Storkøbenhavn, hvor man kan designe sin egen uddannelse, kan i løbet af nogle år blive, at RUC afleverer en fusionsansøgning til Aalborg Universitet. Det ville blandt andet placere det naturvidenskabelige område på RUC mindre marginalt end det er nu. Men jeg synes bedre om visionen om RUC som fagligt brobyggeruniversitet befolket af studerende, der vil selv.

Debatindlægget her blev bragt i Kristeligt Dagblad 13-04-2013 i anledning af lærerkonflikten da.

Tillid fremfor mistillid. Derfor virker den finske skolemodel

13-04-2013

AF HANNE LETH ANDERSEN
OG JENS HØJGAARD JENSEN

DER ER solide grunde til ikke at følge folkerige nationer, når man tilrettelægger skolevæsen og uddannelsessystem. Små nationer har ikke råd til at lade de svageste i stikken og satse på en snæver elite.

Derfor bør Danmark se til Finland snarere end til USA eller Korea. Det er en god idé alene af den grund, at det finske uddannelsessystem er en megasucces – de finske børn klarer sig fremragende.

Det bekymrende er, at man i Finland er imod alt det, politikerne har været i gang med at indføre i Danmark over de seneste 10 år med frit skolevalg, konkurrence og markedskræfter.

I Finland tror man på lighed og solidaritet. I Danmark har vi indført standardiserede test, målstyring og kontrol – i stedet for at lægge vægt på lærernes ekspertise og autoritet. Dette til trods for, at den finske forsker og uddannelsesekspert Pasi Sahlberg i bogen "Finnish Lessons" fra 2011 har dokumenteret, at standardiserede test forringer elevernes udbytte og giver dem dårligere muligheder for at lære. Omtrent samme resultat er forskere fra DPU kommet frem til.

Det er naivt at tro, at fordi test eller fuldt skema virker i Korea eller Canada, så virker det også i Danmark. I Finland bruger man i stedet tiden i skolen på fokuserede læringsaktiviteter med færre timer, kortere dage og mindre hjemmearbejde.

Lærerne har færre timer med klassen og meget mere tid til forberedelse og faglig fordybelse. Der arbejdes dybere i fagene med mindre udenadslære og mere tænkning og analyse. Skolearbejdet tages seriøst og anskues langtfra som en børnepasningsordning.

Grundlæggende kan man anlægge to forskellige tilgange til sine medmennesker. Tillid eller mistillid. Det manifesterer sig radikalt i skole og uddannelse.

Man lægger desuden vægt på lighed. Det betyder, at alle ud over at gå i skole gratis også kan få en god frokost og relevante bøger. Det koster penge, men sikrer opdateret faglig viden og pædagogisk tilgang. Ligheden betyder også, at der ikke er noget separat privat system. Og så betoner man, at lærerne følger eleverne over en årrække, i indskolingsårene, gennem grundskolen og ligeså på mellemtrinnet, for at sikre faglig sammenhæng og personlig indsigt i det enkelte barns udvikling.

Den ultimative lighed består i, at der ikke findes dårlige skoler: Når der ikke findes et parallelsystem for de privilegerede familiers børn, så opstår der heller ikke den stigmatisering af visse skoler, hvor de mindst privilegerede sidder tilbage. Derfor kan forældrene generelt have tillid til skolen.

Det finske system er desuden baseret på en bevidst udvælgelse af de bedst egnede lærerstuderende, som herefter får den bedste forskningsbaserede uddannelse og derefter frie hænder og tillid. De nationale læreplaner er kortfattede, og der er forbud mod standardiserede test frem til 6. klasse og kun meget begrænset anvendelse herefter.

Grundlæggende kan man anlægge to forskellige tilgange til sine medmennesker. Tillid eller mistillid. Det manifesterer sig radikalt i skole og uddannelse.

Danmark har klart befundet sig i den første gruppe, sammen med de andre nordiske lande og haft succes med det. Alligevel arbejder vi os i højt fart hen imod den anden ende af spektret, hvor befolkningsrige lande med mindre tillidskultur og mere elitedyrkelse og ulighed befinder sig.

Hanne Leth Andersen er prorektor og professor i universitetspædagogik, og Jens Højgaard Jensen er lektor i fysik og fysikdidaktik. Begge skribenter er tilknyttet RUC

HVAD MENER DU?

Send din mening til Kristeligt Dagblad eller deltag i debatten på k.dk

Den følgende artikel har sit udspring i min undervisning på kurset Problem Solving in Physics I, som er en del af fysikstudiet på RUC. Og siden den blev skrevet, er den blevet brugt som undervisningsmateriale på kurset.

Introducing fluid dynamics using dimensional analysis

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Many aspects of fluid dynamics can be introduced using dimensional analysis, combined with some basic physical principles. This approach is concise and allows exploration of both the laminar and turbulent limits—including important phenomena that are not normally dealt with when fluid dynamics is first introduced as a precise mathematical discipline. In this paper, we use dimensional analysis to understand drag forces on bodies, flow speeds through tubes, and lift forces.
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I. INTRODUCTION

In this article, we describe a method, using dimensional analysis, for introducing fluid dynamics to undergraduate physics students. Dimensional analysis, which emphasizes thinking more than memorizing, is not commonly used in introductory textbooks[1–6] when discussing fluids, even though such as approach facilitates a clearer understanding of the effects of turbulence when considering drag and lift forces, and flow resistance in tubes.

Fluid dynamics is a huge and difficult subject, with a long history, not only in the field of physics but also in the fields of mathematics and engineering. In spite of this, a complete understanding of turbulence remains an unsolved problem, as several famous physicists have noted.[7]

Turbulence is common in daily life. The transition from laminar to turbulent flow is seen, for example, in smoke rising from a cigarette. Turbulence affects the air resistance acting on a bicyclist[8] and on falling objects such as the coffee filters or muffin forms often used in physics teaching.[9] In these cases, the air resistance is proportional to the square of the object's velocity, and this quadratic dependence on velocity, in incompressible fluids, is connected to turbulence. Nevertheless, in physics instruction drag forces are often assumed to be linear in the velocity (as is the case for laminar flow) rather than quadratic, even though the latter is more commonly encountered.

Within the paradigm of fluid dynamics as an exact mathematical discipline, drag forces proportional to velocity (Stokes' law) and other phenomena (e.g., Poiseuille's law for the flow speed through tubes) in the laminar flow regime can be derived. But due to the nonlinearity of the fundamental equations, this approach does not give results in the turbulent limit. In such situations dimensional analysis is especially useful, in that much of the behavior can be revealed without actually solving the nonlinear equations.

In Sec. II, we give a brief introduction to dimensional analysis, meant for readers unfamiliar with the method. In Secs. III and IV, we estimate drag forces on bodies and flow speeds through tubes in both the turbulent and laminar limits using dimensional analysis combined with work/energy-flow arguments. Instead of comparing the relative strengths of the terms (with dimensions of force) in the Navier-Stokes' equation, as is usually done in introductory textbooks on fluid dynamics, we ask whether the work done by driving forces is converted primarily into thermal energy or primarily into macroscopic kinetic energy. In Sec. V, we calculate lift forces in both the turbulent and laminar limits using dimensional analysis combined with force/momentum-flow arguments. In Sec. VI, we discuss how the usual mathematical way of presenting fluid dynamics has overshadowed the more intuitive approach of Secs. III–V. Finally, in Sec. VII, the pros and cons of the approach are discussed.

II. DIMENSIONAL ANALYSIS

In recognition of the centenary of its introduction, we will use the Bohr model of the hydrogen atom as an example to introduce the method of dimensional analysis.

When presenting his model in 1913, Niels Bohr gave a dimensional analysis argument in the introduction of his article.[10] The classical mechanical problem of calculating the orbits of the electron around the proton involves only two input variables: the mass of the electron and the constant $k_C = e^2/4\pi\epsilon_0$ that determines the strength of the force between the two particles. But these two input variables cannot be combined into a physical quantity with dimensions of length, needed to explain the size of the atom. On the other hand, when Planck's constant is introduced as a third variable, dimensional analysis gives a characteristic length—the Bohr radius—that agrees with the known order of magnitude of the size of the atom. Let us see how.

We stress that the symbols used in physics formulas refer to physical quantities—not merely to the numerical values of these quantities in a particular system of units. For example, if we write $d_1 = 2d_2$, to indicate that one distance is twice as large as another, we are making a statement that is independent of the particular units and numbers used to specify d_1 and d_2.

We also note that it is not possible to add quantities of different dimension—for example, a length and a mass. On the other hand, quantities with different dimensions can be multiplied and divided to obtain quantities with new dimensions (for example, when we calculate a density from a mass divided by a length cubed). In general, if a physical formula expresses some output quantity Q_1 in terms of several input quantities $Q_2, Q_3,\ldots$, then this formula must take the form

$$Q_1 = g \cdot Q_2^{\alpha} \cdot Q_3^{\beta} \cdot Q_4^{\gamma}\ldots, \tag{1}$$

where the powers $\alpha, \beta, \gamma,\ldots$ yield the correct dimensions for Q_1 and the dimensionless factor g can be either a pure number or a function of dimensionless power-law combinations of $Q_2, Q_3, \ldots$, if such dimensionless combinations exist.[11]

Returning to the Bohr atom, we denote the Bohr radius a_0, the mass of the electron m_e, and write the Coulomb force between the electron and the proton as k_C/r^2. We will use square brackets to denote the dimensions of a quantity, and

the symbols L, T, and M to denote the fundamental mechanical dimensions length, time, and mass, respectively. We can then write $[a_0] = L$, $[m_e] = M$, and $[k_C] = M \cdot L^3 \cdot T^{-2}$ (note that the final expression is equivalent to force times L^2).

Now suppose that a_0 were a function of only m_e and k_C. Then, following the form of Eq. (1), we could write

$$a_0 = g \cdot (m_e)^\alpha \cdot (k_C)^\beta \tag{2}$$

for some α and β. Taking the dimensions of both sides of this equation gives

$$L = M^\alpha \cdot (M \cdot L^3 \cdot T^{-2})^\beta = M^{\alpha+\beta} \cdot L^{3\beta} \cdot T^{-2\beta}. \tag{3}$$

Since none of the fundamental dimensions L, T, and M can be written in terms of the others, the powers of each dimension must match up separately between the two sides. But this is impossible—matching powers of L implies $\beta = 1/3$, while matching powers of T implies $\beta = 0$.

Bohr's dimensional analysis thus shows that classical mechanics cannot produce a characteristic atomic length. But including Planck's constant (divided by 2π), which has dimension $[\hbar] = M \cdot L^2 \cdot T^{-1}$, changes the situation. Equations (2) and (3) are then replaced by

$$a_0 = g \cdot (m_e)^\alpha \cdot (k_C)^\beta \cdot \hbar^\gamma \tag{4}$$

and

$$\begin{aligned} L &= M^\alpha \cdot (M \cdot L^3 \cdot T^{-2})^\beta \cdot (M \cdot L^2 \cdot T^{-1})^\gamma \\ &= M^{\alpha+\beta+\gamma} \cdot L^{3\beta+2\gamma} \cdot T^{-2\beta-\gamma}. \end{aligned} \tag{5}$$

Equating the powers of the fundamental dimensions now gives

$$\begin{aligned} M: &\quad 0 = \alpha + \beta + \gamma, \\ L: &\quad 1 = 3\beta + 2\gamma, \\ T: &\quad 0 = -2\beta - \gamma, \end{aligned} \tag{6}$$

and these equations have the unique solution $\alpha = -1$, $\beta = -1$, and $\gamma = 2$. Similarly, searching for combinations of m_e, k_C, and $\hbar$ with dimension $L^0T^0M^0$ gives the unique solution $\alpha = \beta = \gamma = 0$, implying that the factor g in Eq. (4) has to be a pure number. Therefore, from dimensional reasoning we conclude that a theory of atoms built on the input variables m_e, k_C, and $\hbar$, independent of other assumptions, results in a characteristic atomic length given by

$$a_0 = \text{pure number} \cdot \frac{\hbar^2}{m_e k_C}. \tag{7}$$

This expression happens to agree with the textbook formula for the Bohr radius if the pure number is chosen to be 1.

Dimensional analysis thus played a role in the birth of atomic physics and quantum mechanics. Of course, the value (in this case 1) of the pure number cannot be found using dimensional analysis. But aside from this pure number, the Bohr radius can be found using dimensional analysis without reference to a developed model or theory. Any model or theory that uses m_e, k_C and $\hbar$ as input variables will imply Eq. (7). Thus, it is not surprising that the correct theory (Schrödinger's equation) gives the same expression.

It is not common for physicists to follow Niels Bohr and present dimensional analysis in their formal writings. But dimensional analysis is a powerful tool that physicists often use before developing a more formal line of reasoning. Solving problems using dimensional analysis is also a good approach for teaching students how to think like a physicist.[11]

III. DRAG

We now turn to fluid dynamics, and use dimensional analysis to calculate the drag force on a body moving through a fluid. For motion through air, drag is often called air resistance. But the formulas we derive below are valid for the motion of bodies in both gases and liquids. We consider the steady motion of a body pulled though a fluid, as sketched in Fig. 1. The drag D is the force exerted by the fluid in the direction opposite to the motion. If the motion is steady, the drag has the same magnitude as the pulling force.

The first thing to do is to ask which physical quantities the drag can depend on. The pulling force and the drag depend on the velocity v of the body; they also depend on the shape of the body, and on a length r characterizing the size of the body. We further expect the drag to depend on the density ρ of the fluid. And we must expect the drag to depend on the internal "friction" or "stickiness" within the fluid, characterized by its viscosity η.

We might at first think that the mass or density of the body is a relevant physical quantity, but there is no physical mechanism for mass to affect the pulling force unless the body is accelerated. Or we might think that the compressibility of the fluid is relevant—and in fact it is, if v is near to the speed of sound. We therefore restrict ourselves to velocities small compared to the speed of sound, which is equivalent to assuming that the fluid is incompressible. This leaves us with v, r, ρ, η, and dimensionless quantities characterizing the shape of the body, as input variables for D. The formula for D, following Eq. (1), must therefore take the form

$$D = g \cdot v^\alpha \cdot r^\beta \cdot \rho^\gamma \cdot \eta^\delta, \tag{8}$$

with g depending on the shape of the body and possibly on dimensionless combinations of the other variables.

We seek values of α, β, γ, and δ that give the same dimensions on both sides of Eq. (8). To find the dimensions of η, we imagine a plate on the top of a layer of fluid resting on a

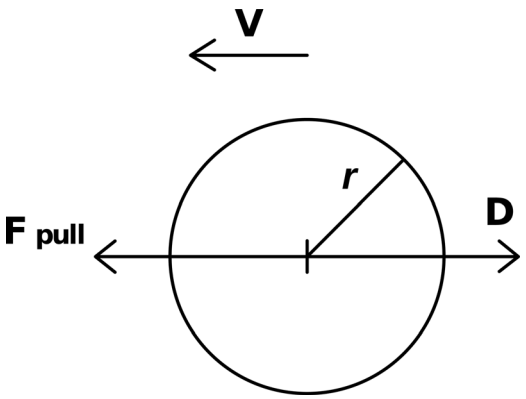

Fig. 1. Drag force on a body pulled through a fluid. Here $\mathbf{F}_{\text{pull}}$ is the pulling force, $\mathbf{D}$ the drag force, $\mathbf{v}$ the velocity of the body, and r a length characterizing the size of the body.

horizontal plane. Acting on the plate with a horizontal force causes the plate to move along with the top of the fluid layer, while the bottom of the layer remains at rest, "stuck" to the surface below. The greater the viscosity, the more force is required, per unit area of the plate, to produce a given velocity gradient (change of velocity per unit height). The viscosity coefficient η is therefore defined as

$$\eta = \frac{\text{force}/\text{area}}{dv/dz}, \tag{9}$$

where z is the vertical direction. From this definition it is straightforward to show that $[\eta] = M \cdot L^{-1} \cdot T^{-1}$.

The dimensions of the remaining quantities in Eq. (8) are familiar, so we can now take the dimensions of both sides of that equation to obtain

$$M^1 \cdot L^1 \cdot T^{-2} = M^{\gamma+\delta} \cdot L^{\alpha+\beta-3\gamma-\delta} \cdot T^{-\alpha-\delta}. \tag{10}$$

Equating the powers of the base dimensions on the two sides of the equation gives

$$\begin{aligned} M: &\quad 1 = \gamma + \delta, \\ L: &\quad 1 = \alpha + \beta - 3\gamma - \delta, \\ T: &\quad -2 = -\alpha - \delta. \end{aligned} \tag{11}$$

Because there are three equations and four unknowns, dimensional analysis does not uniquely determine the formula for D. However, we can use these three equations to eliminate α, β, and γ in Eq. (8), which then becomes

$$D = g \cdot v^{2-\delta} \cdot r^{2-\delta} \cdot \rho^{1-\delta} \cdot \eta^{\delta} = g \cdot \rho r^2 v^2 \cdot \left(\frac{\eta}{\rho r v}\right)^{\delta}, \tag{12}$$

in which only δ and g remain to be determined. The quantity $\eta/\rho r v$ is dimensionless; it is the inverse of Reynolds number $\text{Re} \equiv \rho r v/\eta$.

The unknown g is either a pure number or a function of dimensionless combinations of v, r, ρ, and η. If we modify the preceding analysis to seek a dimensionless combination of these variables, we find that the only possibilities are $(\rho r v/\eta)^{\alpha} = \text{Re}^{\alpha}$. We can therefore rewrite Eq. (12) as

$$D = \rho r^2 v^2 \cdot f(\text{Re}), \tag{13}$$

where f is an unknown function of Re. This is the usual formula for the drag, found in textbooks in fluid dynamics. The function $f(\text{Re})$ depends on the shape of the body and can be very complex. Much effort is invested in understanding this function in fluid dynamics.

Here, however, we will skip the formal reasoning and continue with a more intuitive physical approach. In general, both thermal energy and macroscopic kinetic energy are created when a body is pulled through a fluid. But if we restrict ourselves to either the limit where the velocity of the body is very small, or the limit where the velocity of the body is very large, we can be more explicit.[12] Our guide is the question: Is the work done by the pulling force converted primarily into thermal energy or primarily into macroscopic kinetic energy of the fluid?

If the velocity of the body is sufficiently small, we expect a steady, laminar flow around the body. In the reference frame of the body, the velocity at every location is constant and therefore the macroscopic kinetic energy is constant. This means that the work done by the pulling force must go entirely into thermal energy, caused by the internal friction in the fluid. In this limit, the drag force will therefore depend on the viscosity of the fluid but not on its density, because the density enters into only the macroscopic kinetic energy.

Thus, in the low-velocity limit Eq. (8) is replaced by the simpler formula

$$D = g \cdot v^{\alpha} \cdot r^{\beta} \cdot \eta^{\gamma}, \tag{14}$$

and taking the dimension of both sides gives

$$M^1 \cdot L^1 \cdot T^{-2} = M^{\gamma} \cdot L^{\alpha+\beta-\gamma} \cdot T^{-\alpha-\gamma}, \tag{15}$$

so that Eqs. (11) are replaced by

$$\begin{aligned} M: &\quad 1 = \gamma, \\ L: &\quad 1 = \alpha + \beta - \gamma, \\ T: &\quad -2 = -\alpha - \gamma. \end{aligned} \tag{16}$$

These equations have the unique solution $\alpha = \beta = \gamma = 1$. Moreover, there is no combination of v, r, and η that is dimensionless, so g is simply a number. Our formula for the drag force in this limit therefore becomes

$$D = \text{pure number} \cdot \eta r v, \tag{17}$$

where the pure number depends of the shape of the body. For a sphere of radius r, the number happens to be 6π and Eq. (17) is known as Stokes' law.

In the opposite limit, where the velocity is sufficiently large, we expect a turbulent wake behind the body. When pulling the body through the fluid, we continuously produce new macroscopic kinetic energy in the wake. At sufficiently high velocities, we will assume that the work done by the pulling force is converted predominantly into macroscopic kinetic energy in the wake. The macroscopic kinetic energy of a fluid with a given flow pattern depends of the density of the fluid, but not on its viscosity. Although the viscosity of the fluid determines how the macroscopic kinetic energy in the wake is transformed into thermal energy in the long run, the immediate production of thermal energy in the fluid can be neglected. In this case, we therefore expect the pulling force (and the oppositely directed drag of equal magnitude) to be independent of viscosity, determined only by shape, v, r, and ρ. Again we now have only three input variables, besides the dimensionless shape variables, so the dependence of D on the three variables is determined uniquely. By an argument entirely analogous to those given previously, we find

$$D = \text{pure number} \cdot \rho r^2 v^2. \tag{18}$$

This formula describes the air resistance acting on a bicyclist,[8] a moving automobile,[8] and the falling coffee filters found in school physics experiments.

Comparing Eq. (17), valid for small velocities, with Eq. (13), valid in general, we conclude that $f(\text{Re})$ is proportional to $(\text{Re})^{-1}$ at small velocities. Similarly, by comparing Eq. (18), valid for large velocities, with Eq. (13), we find $f(\text{Re}) = \text{constant}$. Since $f(\text{Re})$ is a function of Re, and not of v, we should rather state that $f(\text{Re}) \propto (\text{Re})^{-1}$ at small values of Re, and that $f(\text{Re})$ is independent of Re at large values of

Re. In the reasoning above r, η, and ρ were assumed constant and only v varied. But the more general statements are that Eq. (17) is valid in the limit of small Reynolds numbers, and Eq. (18) is valid in the limit of large Reynolds numbers. How small or large the Reynolds numbers should be in order to make Eqs. (17) and (18) reliable depends on the shape of the body. In the cases of bicycling, car driving, and falling coffee filters, Eq. (18) is valid and D is thus independent of Re. This means that the assumption of D being independent of η is justified in these cases. In general, at a given Re, Eq. (18) is more valid for not-very-streamlined bodies, where the cross section of the wake is determined by the cross section of the body, not by the detailed streaming of the fluid around the body.

IV. FLOW

Using the same methods as above, we now treat fluid flow through a tube in the two limits. We imagine a fluid being driven through a tube by a pressure difference between its ends, as sketched in Fig. 2. As in the case of drag, we focus on whether thermal energy or macroscopic kinetic energy is created in the tube as the fluid is driven through it.

Our goal is to find a formula for the volume of fluid per unit time flowing through the tube, Q. The flow rate Q must depend on the shape and the characteristic length r of the cross section of the tube. It depends on the pressure difference between the ends of the tube ΔP and on the length of the tube ℓ in the combination $\Delta P/\ell$ because the flow of fluid will be unaltered when doubling both ΔP and ℓ. In general, we expect Q to depend on both the viscosity η and the density ρ of the fluid. As in the preceding section, this leaves one input variable too many to determine the desired formula by dimensional analysis. Thus, we will restrict ourselves to the two limits where the Reynolds number is either very small or very large. The relevant velocity entering the estimate of the Reynolds number ($\mathrm{Re} = \rho r v/\eta$) is the average flow velocity, given by Q divided by the cross-sectional area of the tube.

Consider first the limit where Re is small. Here we expect a laminar and steady flow through the tube. The macroscopic kinetic energy of the fluid is thus the same at the end as at the beginning of the tube. This means that the work done in pressing the fluid through the tube goes entirely into heating the fluid. This heating depends on the fluid's viscosity but not on its density. We thus expect Q to be determined by r, the cross section shape, $\Delta P/\ell$, and η. Because $[Q] = L^3 \cdot T^{-1}$ and $[\Delta P/\ell] = M \cdot L^{-2} \cdot T^{-2}$, the only combination of these quantities that has the correct dimensions is $(\Delta P/\ell) \cdot r^4/\eta$. No dimensionless quantities can be made from $\Delta P/\ell$, r, and η, so the formula for Q at small Reynolds numbers is

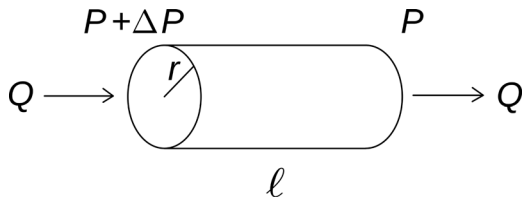

Fig. 2. Fluid flow through a tube due to a pressure difference. Here Q is the volume of fluid driven through the tube per unit time, r a characteristic cross sectional width, ℓ the length of the tube, and P the pressure.

$$Q = \text{pure number} \cdot \left(\frac{\Delta P}{\ell}\right) \cdot \frac{r^4}{\eta}. \tag{19}$$

Inserting $\pi/8$ for the number gives Poiseuille's law for flow through a tube with a circular cross section of radius r.

Next consider the limit where the Reynolds number is large. Here, we expect that the interaction with the walls of the tube causes turbulent motion of the fluid, with increasing velocity components perpendicular to the stream direction as the fluid moves through the tube. The fluid thus leaves the tube with larger macroscopic kinetic energy per unit volume than when it entered. The work done by the pressure difference forcing the fluid through the tube is converted into macroscopic kinetic energy in the fluid, which depends on the fluid's density ρ, but not on its viscosity. In this limit, we therefore expect the flow rate Q to be determined by r, cross section shape, $\Delta P/\ell$, and ρ. Dimensional analysis for the desired formula for Q then results in

$$Q = \text{pure number} \cdot \left(\frac{\Delta P}{\ell} \cdot \frac{r^5}{\rho}\right)^{1/2}. \tag{20}$$

This formula applies for tubes with not-too-smooth internal surfaces, e.g., cast-iron tubes. The formula is seldom derived in physics textbooks; I only found it reported as an empirical result in specialized textbooks in fluid mechanics.[13]

Equation (20) has an interesting consequence. Imagine two containers partly filled with the same fluid connected to each other by a tube at their bottoms. Suppose the surfaces of the fluid in the two containers are initially at different heights, thereafter gradually reaching the same level due to flow, driven by gravity, through the tube. It follows from Eq. (20) that the way this takes place, at large Reynolds numbers, is independent of the fluid: Q is independent of ρ whenever ΔP is proportional to ρ, as in this case for fluids in a gravitational field. Thus, the way potential energy is converted into turbulent macroscopic kinetic energy does not depend on density, just as the conversion of potential energy into kinetic energy during free fall does not depend on mass.

Summarizing, we have derived formulas for both drag and flow, in both the turbulent and the laminar limits. This was accomplished using dimensional analysis combined with physical reasoning about the transformation of work into primarily either thermal energy or macroscopic kinetic energy. In Sec. V, we treat lift forces in general, using dimensional analysis combined with physical reasoning about the flow of momentum in the fluid passing a body.

V. LIFT

Why can dust and sand grains be lifted by the wind, given the fact that they afterward fall to the ground again and thus are not able to float? What is the minimal wind velocity needed to lift the grains from the ground? Let us try to answer this question as an illustration of fluid lift problems in general.

Figure 3 suggests that the lift force L on the grain, due to its interaction with the air passing by the grain, derives from the change of the direction of motion of the air during the passage. According to Newton's third law, the grain must cause a force on the air in the opposite direction and of the same magnitude as the lift force exerted by the air on the grain. According to Newton's second law, an unbalanced

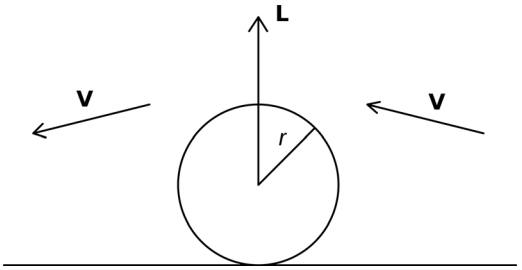

Fig. 3. Lift force on a sand grain. Here **L** is the lift force, **v** the velocity of the wind, and r a characteristic length.

force on the air is accompanied by an equal change in its momentum per unit time. The lift L can thus be evaluated by calculating the vertical momentum change per unit time of the air passing the particle.

Unlike the drag force, the lift force is not intuitively related to friction and the viscosity of the air. We assume that the streaming pattern around the grain is determined by the geometry, the velocity of the air, and the density of the air. Furthermore, we assume that the magnitude of the viscosity influences only the thickness of a thin boundary layer, caused by the condition of zero velocity at the boundary. In this case, the vertical momentum change of the air per unit time depends only on the shape of the grain, the size of the grain r, the density of air ρ_{air}, and the velocity of the wind v. Therefore the lift force, by dimensional analysis, must be

$$L = \text{pure number} \cdot \rho_{\text{air}} \cdot r^2 v^2, \tag{21}$$

where the dimensionless number depends on the shape of the grain.

The grains are lifted from the ground when the lift force exceeds the gravitational force minus the buoyant force. The critical wind velocity v_{crit} for sandstorms is therefore given by

$$\rho_{\text{air}} r^2 v_{\text{crit}}^2 = \text{pure number} \cdot (\rho_{\text{sand}} - \rho_{\text{air}}) r^3 g \tag{22}$$

or

$$v_{\text{crit}} = \text{pure number} \cdot \sqrt{rg\left(\frac{\rho_{\text{sand}}}{\rho_{\text{air}}} - 1\right)}, \tag{23}$$

where ρ_{sand} is the density of sand and g is the strength of the gravitational field. The static buoyant force is included so that Eq. (23) also covers phenomena such as streaming water lifting sediments in rivers and the sea, where ρ_{water} cannot be neglected compared to ρ_{sand}. Equation (23) is found in the literature on dynamics of sand, wind, and barchans.[14]

After being lifted, the grain has a symmetric stream pattern around it, with no resulting vertical momentum transfer between it and the air. The grain will thus fall to the ground again because of the absence of a lift force.

It is worth emphasizing that we assumed neither laminar nor turbulent flow when deriving Eq. (21). Contrary to the expressions for drag and flow, the same formula for lift applies in both the low-velocity and high-velocity limits.

VI. MATHEMATICS OVERSHADOWING PHYSICAL INTUITION

The discipline of fluid dynamics has a long and impressive history. The master's thesis of Werner Heisenberg in 1923, at the Institute of Theoretical Physics in Munich, was in fluid dynamics with Arnold Sommerfeld as his supervisor. Fluid dynamics was considered an important part of theoretical physics in Germany at that time. In England the subject was studied in departments of applied mathematics. Whether fluid dynamics was characterized as theoretical physics or applied mathematics, it was approached as an exact mathematical discipline. Thus the first edition from 1879 of Horace Lamb's classical book about hydrodynamics[15] was entitled *Treatise on the Mathematical Theory of the Motion of Fluids*. It is a highly mathematical book, even though it was renamed *Hydrodynamics* in later editions.

In *The Feynman Lectures on Physics*, Richard P. Feynman[16] divides fluid dynamics into "The Flow of Dry Water" and "The Flow of Wet Water." Dry water is his name for ideal fluids with no viscosity; wet water is the name for real fluids with viscosity. According to Feynman, "John von Neumann was well aware of the tremendous difference between what happens when you do not have the viscous terms and when you do, and he was also aware that, during most of the development of hydrodynamics until about 1900, almost the main interest was in solving beautiful *mathematical* problems with this approximation which had almost nothing to do with real fluids. He characterized the theorist who made such analyses as a man who studied 'dry water.'" The term dry water is thus originally due to von Neumann.

In his introduction to the flow of wet water, Feynman nevertheless follows the tradition and puts the emphasis on mathematical beauty. He ends up with the Navier-Stokes equation for an incompressible fluid in the form

$$\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times \mathbf{u}) = \frac{1}{\text{Re}} \cdot \nabla^2 \Omega; \quad \Omega = \nabla \times \mathbf{u}. \tag{24}$$

Here distances, velocities, and time are measured in units of r, v and r/v, with r being a characteristic length and v a characteristic velocity of a problem considered, Re being the Reynolds number. Equation (24) is not only a beautiful mathematical result but also very useful because it proves that geometrically similar problems have the same relative velocity field **u**, as long as their Reynolds numbers are identical. This theorem justifies applying the results of wind-tunnel measurements on small-scale airplanes or model-basin results on scale-model boats to full-scale objects.

For both mathematical and engineering reasons, the tradition in fluid mechanics is therefore to consider different physical quantities as functions of the Reynolds number. The drag force on a body pulled through a fluid is in general expressed in the form of Eq. (13). The function $f(\text{Re})$ is called the drag coefficient. In most textbooks in fluid mechanics, empirically found drag coefficients for different shapes are presented as functions of Re. In the limit of small Reynolds numbers—that is, the limit of small velocities if r, ρ, and η are held fixed—$f(\text{Re})$ is found to be inversely proportional to Re, giving Eq. (17), Stokes' law.

In the limit of large Reynolds numbers (large velocities), it is, however, not customary to report $f(\text{Re})$ to be a constant function, independent of Re, as in Eq. (18). Lautrup has written an excellent book introducing fluid dynamics within the mathematical tradition, where he merely mentions that the drag becomes independent of the Reynolds number at large Reynolds numbers.[17] Landau and Lifshitz argue that the drag

coefficient for a flat disk held perpendicular to the flow direction must be independent of Re in the high velocity limit, but this is again mentioned only as an aside.[18] In general, the focus in the literature is on Eq. (13) with its four input variables, not allowing D to be determined by dimensional analysis. Authors rarely give arguments for D being independent of viscosity at high velocities and therefore having only three input variables and being given by dimensional analysis as Eq. (18).[19] As mentioned, this leaves many daily experiences unexplained.

What is the explanation for the lack of attention to drag forces proportional to v^2, despite their frequent appearance in nature and in everyday life? The best explanation I can give is that the prevailing paradigm in fluid mechanics, developed by mathematicians on the theoretical side, and engineers on the practical side, expresses everything as a function of the Reynolds number, which overshadows the more intuitive physical arguments leading to Eq. (18).

Likewise, when dealing with flow through a pipe, the paradigm seems to have overshadowed the arguments leading to Eq. (20).

VII. DISCUSSION

Of course the high-velocity limit with turbulence—or more correctly, the limit of high Reynolds numbers—is difficult to tackle compared to the limit of small velocities and laminar flow. Experience shows that laminar flow, and consequently the expressions in Eqs. (17) and (19) for D and Q, is always reached eventually as the velocity is reduced. However, the expressions in Eqs. (18) and (20) for D and Q cannot always be reached when the velocity is increased, as we shall now see.

First consider drag. Here, Eq. (18) for a flat disk perpendicular to the direction of the flow is usually reached at not-too-large velocities, long before the assumption of incompressibility breaks down at velocities comparable to the speed of sound. But this is not the case for very streamlined objects such as wings of birds and airplanes. What is wrong with the argument in Sec. III, leading to Eq. (18)? The argument was that the work done by the pulling force to move a body through a fluid, in the high-velocity limit, results mainly in macroscopic kinetic energy in the turbulent wake behind the body. And because the macroscopic kinetic energy depends on ρ, not on η, the same must be true for the pulling force and thus the drag.

The shortcoming of this argument is that, although only ρ is needed to express the macroscopic kinetic energy in the wake, the macroscopic kinetic energy in the wake also depends on η because η determines how the wake is created at the body. In a fluid without viscosity there would not be a turbulent wake. For many sorts of bodies the influence of η is so complicated that the drag force, over a certain range of the Reynolds number, decreases with increasing velocity, because of a narrowing of the wake with increasing velocity.

The circumstance where the argument leading to Eq. (18) is justified is when the turbulent wake is determined by geometry alone, as in the case of the flat disk perpendicular to the flow. Although viscosity in the fluid is here also needed to create a turbulent wake, the magnitude of the viscosity does not matter if η is larger than a certain minimum size. An analog is a problem involving static friction. A minimal static friction coefficient may be needed, but if this condition is satisfied, the solution does not depend on the actual value of the friction coefficient. As mentioned, many daily phenomena act more like the flat disk than like a streamlined body: turbulent wakes are, in essence, determined by the cross section to the flow, not by the magnitude of the viscosity.

For flow through a pipe, Eq. (20) has been experimentally verified for rough pipes, but not for smooth pipes.[13] The less rough the pipe is, the higher is the Reynolds number that must be exceeded before the formation of turbulence is independent of the viscosity, corresponding to Eq. (20).

In the derivations of Eqs. (19) and (20), the pressure gradient was assumed constant along the pipe. The quantities ΔP and ℓ appeared only in the combination $\Delta P/\ell$ in the dimensional analysis. Due to translational symmetry, this assumption is valid in the laminar limit. However, in the turbulent limit, with increasing turbulent macroscopic kinetic energy along the pipe, the validity of this assumption is not obvious. In practice, it seems to be a valid approximation.[13] Theoretically, an unknown function of ℓ/r would have to be substituted for the pure number in Eq. (20) if ΔP and ℓ are considered independent input variables.

Explanations of lift on airplanes were discussed some time ago in this journal by Weltner.[20] He argued that an explanation based on repulsive forces is easier to assimilate, more powerful, and better related to the basic principles of mechanics than the frequently used explanation using Bernoulli's equation. Furthermore, he argued that the explanation based on Bernoulli's equation has fundamental drawbacks and is based on incomplete reasoning. Weltner argued "that the high streaming velocity at the upper side of the aerofoil is not the reason for the low pressure. To the contrary, the low pressure generated by the aerofoil is the reason for the high streaming velocity."

The treatment of lift in Sec. V, which relates the lift to repulsive forces and not to Bernoulli's equation, is in accordance with Weltner. Equation (21) is relevant not only for the lift on sand grains but also for the lift on airplanes, even though an airplane wing, unlike a sand grain, is not characterized by just one length. If we assume that the flow pattern around the wing is given by geometry and is independent of the value of the viscosity of the air, dimensional reasoning tells us again that ρ_{air} and v go into the expression for the lift with the factor $\rho_{\text{air}}v^2$, as in Eq. (21).

The assumption of a solely geometrically determined flow pattern is common. It was also made by Weltner for calculating the momentum transfer to the air from a steady laminar flow around an airplane wing. The assumption is, however, equally valid when the flow is turbulent. In both the turbulent and the laminar limits, the value of viscosity merely affects the thickness of a boundary layer, which can be treated as a part of the body to be lifted. Between these two limits, the ways in which vortices go into the flow depend in general on the values of the viscosity and the Reynolds number. Thus, the number in Eq. (21) is not merely a constant but a function Reynolds number. It is only in the two limits that dimensional analysis leads to a v^2 dependence of the lift.

In summary, drag, flow, and lift can all be explained qualitatively in an introductory physics course by relating these phenomena to the basic principles of physics. In the case of lift forces, the basic principles are Newton's second and third laws. For drag forces and flow through a pipe, the basic principle is the work-energy theorem, with the energy produced

becoming primarily either macroscopic kinetic energy or thermal energy. Through the use of dimensional analysis these explanations can be developed to give quantitative results, both in the laminar and the turbulent limits. Because the results in the turbulent limit, not usually presented in physics textbooks, are in accordance with many daily experiences, examining such cases via dimensional analysis provides a good introduction to fluid dynamics before drowning in the mathematical details and intermediate-Reynolds-number phenomena relevant for more-or-less streamlined bodies and smooth tubes.

VIII. CONCLUDING REMARKS

The mathematical and experimental complexities of the huge subject of fluid dynamics have, in the physics teaching tradition, led to two alternatives: either the subject is mostly avoided or it is taught as an exact mathematical discipline. In both cases, common phenomena such as drag forces proportional to the velocity squared are not discussed.

Fortunately, the observation that mathematics can cast a shadow over physical intuition is not common in physics. In fact, mathematical thinking has been a characteristic aspect of physics for 400 years, ever since Galileo stated that the book of nature is written in the language of mathematics. In particle mechanics, for instance, precise mathematical thinking may be essential.[21] Nevertheless, beginning with Maxwell the letter symbols in the mathematical equations of physics have gradually come to be understood as representing physical quantities, not merely pure numbers, thus transforming theoretical physics from being understood as number calculus to quantity calculus.[22] This development has made dimensional analysis another important aspect of theoretical physics.

We conclude by noting that dimensional analysis is a powerful tool in physics; introducing fluid dynamics through dimensional analysis, combined with basic physical principles, makes it possible in an introductory physics course to deal with drag, flow, and lift, in both the laminar and turbulent limits.

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Forfatterne til artiklen i Physica Scripta, som det følgende er en kommentar til, besvarede min kommentar. Det var for mig interessant, at forløbet illustrerede, at formler blandt fysikere kan opfattes forskelligt. Hvor "teoretikere" forstår dem som udtryk for sammenhænge imellem fysiske størrelser som konsekvens af bogstavregning ud fra i forvejen givne formler og antagelser i forhold til det, der skal findes en formel for, forstår "eksperimentalister" i højere grad formler som noget, der udtrykker talmæssige sammenhænge imellem målinger. Normalt giver disse to opfattelser ikke anledning til uenighed om det faktiske. Men i tilfældet her gjorde det.

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Comment on 'The motion of an arbitrarily rotating spherical projectile and its application to ball games'

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Abstract

In a recent paper (Robinson G and Robinson I 2013 *Phys. Scr.* **88** 018101) the authors developed the differential equations which govern the motion of a spherical projectile rotating about an arbitrary axis in the presence of an arbitrary wind, assuming that both the drag force and the lift force are independent of the Reynolds number and proportional to the square of the projectile's velocity. In this paper, by dimensional analysis, the latter assumption is shown to be incorrect for forces dependent on the angular velocity of the projectile, e.g. the lift force.

Keywords: dimensional analysis, motion of rotating spheres, drag force, lift force

1. Introduction

In a recent paper (Robinson and Robinson 2013) the differential equations governing the motion of a spherical projectile rotating about an arbitrary axis in the presence of an arbitrary wind are developed. Integrating the equations, the authors derive different types of motion experienced in ball games. The paper opens up a fascinating arena for students interested in ball game practice, on the one hand, and in physics and numerical solution of differential equations, on the other. However, dimensional analysis shows that some of the basic assumptions in the paper are incorrect.

Mathematical thinking has been a characteristic aspect of physics for 400 years, ever since Galileo stated that the book of nature is written in the language of mathematics. During the last 100 years, the letter symbols in the mathematical equations of physics have gradually come to be understood as representing physical quantities, not merely pure numbers, thus transforming theoretical physics from being understood as 'number calculus' to 'quantity calculus' (see, e.g., de Boer 1995). This development has made dimensionless analysis an important aspect of theoretical physics in addition to mathematical thinking. In the following section I give my preferred way of briefly introducing dimensional analysis (see also Jensen 2013). For further introduction to dimensional analysis, see for example (Misic *et al* 2010) and the classic text (Bridgman 1922).

In this comment we restrict ourselves to a discussion of the basic assumptions in the paper of Robinson and Robinson (2013), using dimensional analysis. Numerical integration of corrected equations of motion has to be carried out in order to evaluate the consequences of the points made here for the results derived by Robinson and Robinson (2013; hereafter R&R).

In equation (3) of R&R the drag force from the air on the projectile is given as

$$F_{\mathrm{D}} = \tfrac{1}{2}\rho A C_{\mathrm{D}} V^2. \tag{1}$$

In equation (5) of R&R the lift force from the air on the projectile is given as

$$F_{\mathrm{L}} = \tfrac{1}{2}\rho A C_{\mathrm{L}} V^2. \tag{2}$$

In both expressions ρ is the density of the air, A the cross section area of the projectile and V the velocity of the projectile. The dimensionless coefficients C_{D} and C_{L} are called the drag coefficient and lift coefficient, respectively. The basic assumption made by R&R is that both C_{D} and C_{L} are independent of the Reynolds number and independent of V. For

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dimensional reasons this is not possible if the coefficients depend on the angular velocity, ω, of the rotation of the projectile, which certainly is the case for C_L, and possibly for C_D.

2. Dimensional analysis

We use square brackets to denote the dimensions of a quantity, and the symbols L, T, and M for the fundamental dimensions length, time and mass, respectively. Following this notation, $[\rho AV^2] = ML^{-3} \cdot L^2 \cdot L^2T^{-2} = MLT^{-2} = [F_D] = [F_L]$ proves that C_D and C_L are both dimensionless. In this section we discuss C_D and C_L simultaneously, giving them the common name C.

In order to develop a general formula for C we ask which physical quantities C can depend on in an arbitrary medium. The physical quantities of relevance are R, V, ω, ρ, η, and θ, the radius, velocity and angular velocity of the projectile, the density and viscosity of the medium, and the angle between the directions of velocity and angular velocity. Assuming that the medium is incompressible, i.e. V is small compared to the velocity of sound, one can hardly imagine other input variables of relevance for the size of C.

In general, if a physical formula expresses some output quantity Q_1 in terms of several input quantities Q_2, Q_3, … then this formula must take the form

$$Q_1 = g \cdot (Q_2)^{\alpha} \cdot (Q_3)^{\beta} \cdot (Q_4)^{\gamma} \ldots, \tag{3}$$

where the powers α, β, γ, … yield the correct dimensions for Q_1 and g is a dimensionless factor which can be either a pure number or a function of dimensionless power-law combinations of Q_2, Q_3, … if such combinations exist (Jensen 2013). This is the case when the symbols used in physics formulas refer to physical quantities—not merely to numerical values of these quantities in a particular system of units. In that case, it is not possible to add quantities of different dimensions—for example, a length and a mass. On the other hand, quantities with different dimensions can be multiplied and divided in order to obtain quantities with new dimensions (for example, calculating a density from a mass divided by a length cubed). Equation (3) is a consequence of this. It may be possible to choose the powers α, β, γ, … in different ways, resulting in different physical quantities, A, B, …, all having the dimensions for Q_1, allowing Q_1 to be given the form g times a linear combination of A, B, ….. But then, since $aA + bB + \ldots = A(a + bB/A + \ldots)$, we can regain the form given in equation (3) by including the dimensionless parenthesis in g. Thus, equation (3) represents all possible physical formula for $Q_1(Q_2, Q_3, \ldots.)$.

Using equation (3) for C as Q_1 is a special case, since C is dimensionless. However, we will use it anyway in order to illustrate the method of dimensional analysis in general. Following equation (3), the formula for C must take the form

$$C = g \cdot R^{\alpha} \cdot V^{\beta} \cdot \omega^{\gamma} \cdot \rho^{\delta} \cdot \eta^{\varepsilon} \cdot \theta^{\zeta}. \tag{4}$$

Noting that $[C] = M^0L^0T^0$, $[R] = L$, $[V] = LT^{-1}$, $[\omega] = T^{-1}$, $[\rho] = ML^{-3}$, $[\eta] = ML^{-1}T^{-1}$, and $[\theta] = M^0L^0T^0$, the dimensional claim $[C] = [R\alpha V^{\beta}\omega\gamma\rho^{\delta}\eta^{\varepsilon}\theta^{\zeta}]$ implies that

$$\begin{aligned} M&: 0 = \delta + \varepsilon \\ L&: 0 = \alpha + \beta - 3\delta - \varepsilon \\ T&: 0 = -\beta - \gamma - \varepsilon, \end{aligned} \tag{5}$$

since none of the fundamental dimensions M, L or T can be written in terms of the others, and the powers of each dimension consequently must match up separately between the two sides. From these three equations α, β and δ can be determined in terms of γ and ε. From the first equation we get $\delta = -\varepsilon$, from the third $\beta = -\gamma - \varepsilon$, which in combination with the second gives $\alpha = \gamma - \varepsilon$. Dimensional analysis thus restricts the product function in equation (4) to take the form

$$\begin{aligned} &R^{\gamma-\varepsilon} \cdot V^{-\gamma-\varepsilon} \cdot \omega^{\gamma} \cdot \rho^{-\varepsilon} \cdot \eta^{\varepsilon} \cdot \theta^{\zeta} \\ &\quad = (R\omega/V)^{\gamma} \cdot (\eta/RV\rho)^{\varepsilon} \cdot \theta^{\zeta}, \end{aligned} \tag{6}$$

where γ, ε and ζ are arbitrary numbers. The analysis also proves that the function g in equation (4) can be expressed as a function of only the three dimensionless quantities $R\omega/V$, $\eta/RV\rho$ and θ, since equation (6) proves that all dimensionless power-law combinations of the input quantities R, V, ω, ρ, η, and θ can be derived as power-law combinations of just those three. Consequently C can be described as a function of $R\omega/V$, $\eta/RV\rho$, and θ, with $\eta/RV\rho \equiv$ Reynolds number$^{-1} \equiv$ Re^{-1}.

Thus, assuming that C is independent of Re, any dependence of ω has to be via a dependence of $R\omega/V$. Assuming that C is independent of Re, but dependent of ω without being dependent of V, is not a possible assumption.

3. Conclusions

Dimensional analysis does not immediately affect the expression for C_D used by R&R. In ball games the velocities are typically large enough that the drag can be considered independent of the Reynolds number. With reference to observational data, R&R assume C_D to be a constant (= 0.45) in the relevant interval of Re. With no dependence on ω, a drag force proportional to the square of the projectile follows. However, R&R also refer to experimental results where ω is found to affect C_D (see e.g. their equation (4)). In that case, assuming C_D is independent of Re, C_D is necessarily dependent on V, and thus F_D cannot be simply proportional to V^2.

Since, in contrast to C_D, $C_L = 0$ for $\omega = 0$, C_L's dependence on ω cannot be neglected. R&R use an experimentally found relation between C_L and ω in their equation (10):

$$C_L = 3.19 \cdot 10^{-1}\,[1 - \exp(-2.48 \cdot 10^{-3}\omega)], \tag{7}$$

where ω is in rad s^{-1}. However, they assume that C_L is both independent of Re and V, which contradicts the dimensional analysis. If C_L is independent of Re and dependent on ω, C_L is also dependent on V, and consequently F_L is not

proportional to V^2. According to R&R, $V = 115\ \mathrm{km\ h^{-1}}$ and $R = 0.84$ inches in the measurements leading to equation (7). Using these values, equation (7) should instead be substituted by the equation

$$C_L = 3.19 \cdot 10^{-1}\,[1 - \exp(-3.7 \cdot (R\omega/V))]. \quad (8)$$

At small values of $R\omega/V$ we can in general use a linear approximation for C_L. Including the θ dependence and assuming that C_L is independent of Re, we get

$$C_L(R\omega/V, \mathrm{Re}, \theta) = R\omega/V \cdot f(\theta), \quad (9)$$

where $f(\theta)$ is an unknown function of θ. Insertion into equation (2) finally gives

$$F_L = \tfrac{1}{2}\rho A R\omega V \cdot f(\theta). \quad (10)$$

The lift force is thus forced to be proportional to V and not to V^2 as assumed by R&R.

This proportionality can also be derived using Bernoulli's equation for a flow like that pictured in figure 1 of R&R. The flow in a non-rotating frame of reference following the sphere is assumed to be a superposition of a flow created by a rotating sphere without translational motion of the air and a flow around a non-rotating sphere in translational moving air. The pressure difference between the high pressure region and the low pressure region in the flow around the rotating sphere is then, according to Bernoulli's equation, proportional to $\tfrac{1}{2}\rho(V + R\omega)^2 - \tfrac{1}{2}\rho(V - R\omega)^2 = 2\rho R\omega V$. Thus, the lift force, being more or less the cross-sectional area of the sphere times this pressure difference, $F_L \approx \rho A R\omega V$, is seen to be in accordance with equation (10).

Since C_L varies with ω, the above dimensional analysis shows that C_L depends on V if C_L is assumed to be independent of Re. R&R explicitly assume C_D to be independent of Re. Concerning C_L, their assumption of independence of Re is indirect by their use of equation (7) (their equation (10)), independent of Re. It is dimensionally possible to maintain the assumption of C_L being independent of V. But then, instead of being a function of $R\omega/V$ and θ, C_L must be a function of $R^2\rho\omega/\eta$ (= $\mathrm{Re} \cdot R\omega/V$) and θ. Since C_L, besides being described as a function of the three arguments $R\omega/V$, Re and θ, for dimensional reasons can equally well be described as a function of the three arguments $R\omega/V$, $R^2\rho\omega/\eta$ and θ, we may also reason that independence of V in this representation of the function means independence of $R\omega/V$. Thus, C_L only depends on $R^2\rho\omega/\eta$, besides θ. Therefore, using the η/ρ value $1{,}52 \cdot 10^{-5}\ \mathrm{m^2\ sec^{-1}}$ for air at room temperature and $R = 0.84$ inches from the measurements leading to equation (7), equation (7) should be substituted by the equation

$$C_L = 3.19 \cdot 10^{-1}\,[1 - \exp(-8.2 \cdot 10^{-5} \cdot (R^2\rho\omega/\eta))]. \quad (11)$$

Instead of substituting equation (7) by equation (8) when assuming C_L to be independent of Re, we must make this substitution when assuming C_L to be independent of V. Thus, the assumption that C_L is independent of V, besides evaluating it directly, may also be evaluated indirectly by investigating its dependence on, e.g. R.

Theoretically, C_D and C_L may also be described as a function of $R\omega/V$, $\eta\omega/(\rho V^2)$ [= $R\omega/V \cdot \mathrm{Re}^{-1}$] and θ. This choice of arguments shows, like the set $R\omega/V$, Re and θ, that theoretical independence of η or ρ leads to equation (8). It also proves that assuming independence of R implies dependence of only $\eta\omega/(\rho V^2)$, and θ. As we have seen, different interrelations between the independence and dependence of various physical quantities may be investigated choosing different sets of arguments. In practice independence of an argument means weak dependence of the argument compared to the dependence of the other argument, besides θ. If no independence is assumed, dimensional analysis only implies that C_D and C_L as functions of the six input quantities R, V, ω, ρ, η and θ are restricted to be functions of *three* input quantities, e.g. ($R\omega/V$, Re, θ).

We look forward to seeing more experimental results and further simulations of relevant differential equations to deepen our understanding of these fascinating phenomena.

Acknowledgments

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KOMPETENCEBESKRIVELSER – et middel til styring og kontrol eller et middel til styrkelse af faglig autonomi?

1.DJØFOKRATIET – fra folkestyre til folkestyring. Kompetencebeskrivelser indgår i kampen for eller imod denne udvikling. Fortolkningsretten til at afgøre, hvad der er gode/dårlige kompetencebeskrivelser, er derfor vigtig. Evner til at levere relevante nuanceringer giver adgang til fortolkningsretten.

2. KOMPETENCEBESKRIVELSER til INTERNT eller EXTERNT brug.

INTERNt orienteret kompetencebeskrivelse af matematik (fra "Kompetencer og matematiklæring", 2002):

I At **spørge** og **svare** i, med, om matematik

1. **Tankegangskompetence**
2. **Problembehandlingskompetence**
3. **Modelleringskompetence**
4. **Ræsonnementskompetence**

II At omgås **sprog** og **redskaber** i matematik

5. **Repræsentationskompetence**
6. **Symbol- og formalismekompetence**
7. **Kommunikationskompetence**
8. **Hjælpemiddelskompetence**

III **Overblik** og **dømmekraft** vedrørende matematik som fagområde

1. **Matematikkens faktiske anvendelser i andre fag- og praksisområder**
2. **Matematikkens historiske udv., såvel internt som i samf. belysning**
3. **Matematikkens karakter som fagområde**

EKSTERNt orienteret kompetencebeskrivelse af matematik (fra Jens Højgaard Jensen, 2007):

I At kunne matematik

II At kunne bringe matematik i anvendelse

III At kunne vurdere anvendelser af matematik

3. FORSKELLIGARTEDE KOMPETENCER:

I Kompetencer baseret på tilegnede anskuelsesmåder

II Kompetencer baseret på tilegnede færdigheder

III Kompetencer baseret på tilegnet viden og lærdom

4. FORSKELLIGARTEDE PROBLEMER at imødegå med kompetencebeskrivelser:

I **PENSUMITIS** (ved **internt** orienterede kompetencebeskrivelser)

II **BEHAVIORITIS** (ved **eksternt** orienterede kompetencebeskrivelser)

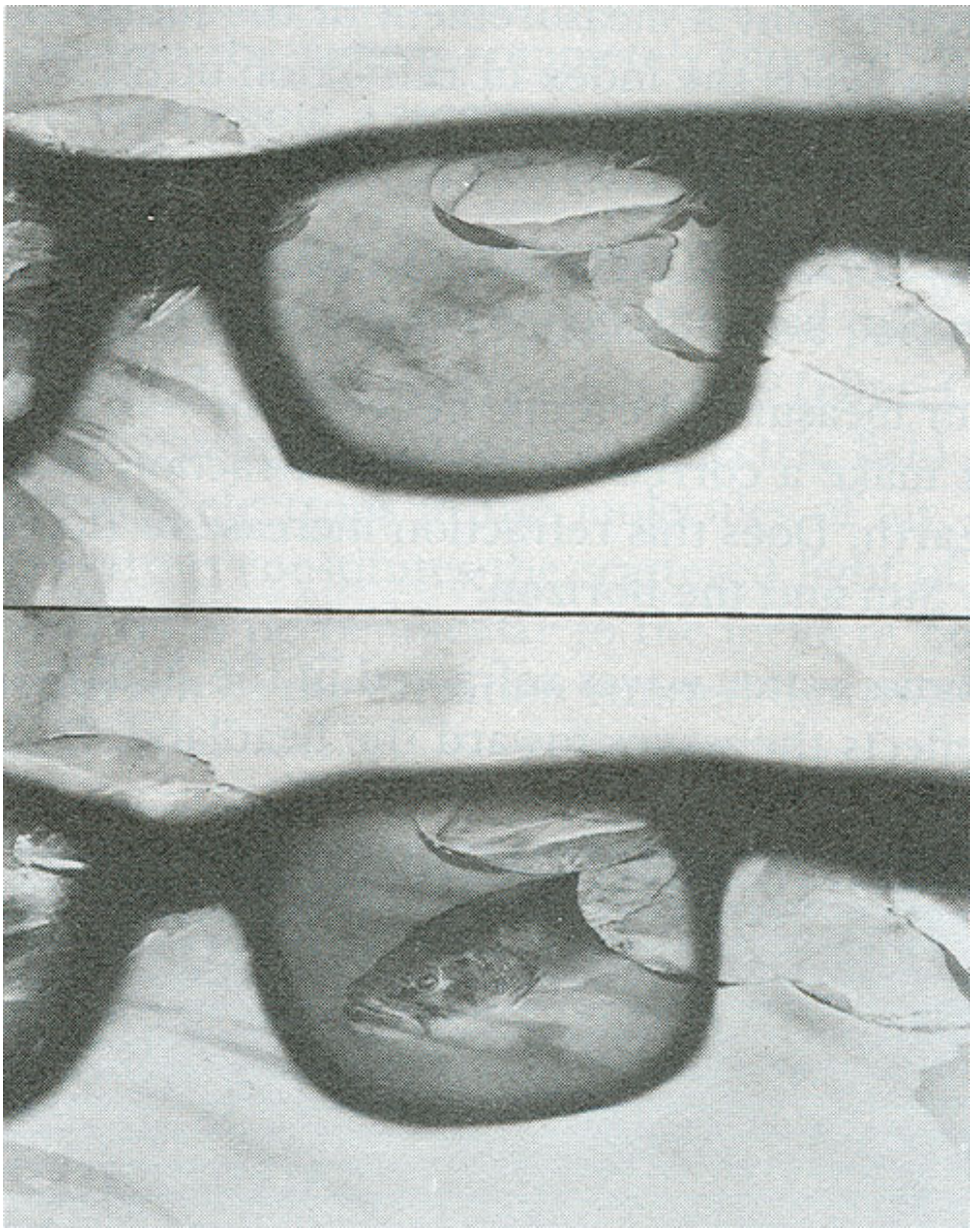

Fig. 37.32 When viewed through ordinary sunglasses (top), the fish is almost completely hidden by the glare reflected by the water. When viewed through Polaroid sunglasses (bottom), the fish becomes visible because the glare is reduced.

“At se er at se bort fra”

Den følgende artikel samler en diskussion op, som vi, i kraft af at være et blandet miljø af matematik og fysik, har haft i mange år i IMFUFA.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY, 2016
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Problem solving in the borderland between mathematics and physics

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ABSTRACT
The article addresses the problématique of where mathematization is taught in the educational system, and who teaches it. Mathematization is usually not a part of mathematics programs at the upper secondary level, but we argue that physics teaching has something to offer in this respect, if it focuses on solving so-called unformalized problems, where a major challenge is to formalize the problems in mathematics and physics terms. We analyse four concrete examples of unformalized problems for which the formalization involves different order of mathematization and applying physics to the problem, but all require mathematization. The analysis leads to the formulation of a model by which we attempt to capture the important steps of the process of solving unformalized problems by means of mathematization and physicalization.

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Mathematics education; physics education; problem solving; mathematization; physicalization

SUBJECT CLASSIFICATION CODES
97D30; 97M50

1. Introduction

Mathematics is the largest teaching subject in the world. Why? Is it to comply with pupils', students', and society's need for sharpened stringent ability to think? Or is it to comply with pupils', students', and society's need to bring mathematics in play in connection to modelling and problem solving – that is to *mathematize*? Both, of course! But if the teaching is managed by teachers trained in 'pure mathematics' rather than 'applied mathematics' and if they see themselves as representing the pure rather than the applied tradition, it is only the first of these two needs which may be obviously fulfilled. Even though pure and applied mathematics share many technical features, the epistemologies of the two subjects are quite different. The notion of counter examples, for example, has very different meanings in pure and applied mathematics, respectively. While a counter example in pure mathematics may wreck all hope for a given hypothesis, this is not necessarily the case in applied mathematics where it may only lead to a revision of the idealization of the given extra-mathematical situation to be modelled.

Applying mathematics in modelling and problem-solving situations is a competence, and this *mathematization* competence is completely different from being able to think mathematically stringent. The question is, therefore, if society and the students obtain what

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they are actually expected to from the present mathematics teaching, or if they get something else? If the latter is the case, the next question becomes: what is required in order to deliver what was originally expected in relation to train students in being able to apply mathematics in the extra-mathematical domain?

A substantial part of doing physics is about applying mathematics when describing nature. Hence, the question is if physics as practicing terrain for learning to apply mathematics to extra-mathematical situations, due to the evolution of the subjects, has become a decisive potential relation between the two teaching subjects. In some physical theories, mathematics and physics are so integrated that viewed historically some of the theories, such as hydrodynamics, have been regarded as theoretical physics at some places (in Germany in the beginning of the twentieth century) and as applied mathematics at other places (in England at the same time). At any rate, the theories differ decisively from pure mathematics: while such theories concern mathematical modelling of existing phenomena, pure mathematics concerns the development and investigation of logical structures. However, it also differs crucially from the experimental and phenomenon-oriented physics. The latter approach is characteristic of much physics teaching at the primary and lower secondary level; consequently, the potential of physics as a training ground for mathematization is rarely realized at these levels.

Hence, mathematization in relation to problem solving is a process which sometimes falls between two stools, or two educational programs to be precise. To some extent this is the case for the classical upper secondary teaching in Denmark. In the tasks on the national, written test, the mathematization process is most commonly provided to the students, and what they have to do is to de-mathematize, i.e. interpret the mathematical result obtained.[1] The oral exams typically focus on mathematical concepts and the reproduction of known proofs. So, if the teachers teach to the test, then mathematization does not occupy a large role in the mathematics program.

Furthermore, physics teaching and physics teachers are typically oriented towards experimental investigations and theoretical understanding of natural phenomena and laws of nature, rather than problem solving. More generally, within the mathematics and physics programs and in the didactics of the disciplines, the division between mathematics and subjects involving applied mathematics appears to stand in the way for a deliberate effort of opening up the mathematics-oriented tasks, where the main challenge is a matter of mathematization, i.e. the translation of the task into the realm of mathematics, rather than mathematical analysis within the mathematical universe. Hence, one might ask: if the mathematics teacher teaches mathematical techniques and proving, and the physics teacher teaches natural phenomena and laws of nature, who then teaches mathematization? Who takes care of the *borderland* between mathematics and physics – a place of potentially greater societal importance than both mathematics and physics in the narrow sense?

At Roskilde University, we have experience with teaching in the borderland between mathematics and physics in a comprehensive course in physics at the bachelor level. This course[2] has the training of mathematization and other forms of formalization abilities as a central goal. The present title of the course is 'Problem Solving in Physics.' Below we solve and comment on four of the problems on which the course is based. These problems are – as well as the other problems in the course – *unformalized problems*, which are formulated in everyday language, meaning that a major challenge for the problem solver is to formalize them, i.e. take a problem from the non-physics context and formulate it in mathematical

and/or physical terms. The four problems are taken from a collection of about 700 such problems.[3] The selected problems illustrate that mathematization often is the crucial part of the formalization when solving unformalized problems. In our experience the formalization, in particular the mathematization, is the part of the solution process of the problems that is the greatest barrier when students engage with such problems. Hence, it is reasonable to suggest – and this is our message with the present article – that physics education (or education involving other disciplines in mathematical guise) based on such unformalized problems can be used as a training ground for teaching mathematization. Although some of the scenarios to be presented do draw upon an arsenal of mathematics and may seem a bit troublesome at first glance, we encourage our readers to stay with us, since each example is carefully chosen to illustrate a point.

We would like to clarify that the point of this article is not to argue – as some do – that physics concepts have a role play in providing students with an understanding of mathematical concepts and ideas grounded in real examples (e.g. speed as a concrete instance of derivative). Rather, the point is that physics can be used to assist students in learning how to apply mathematics to the real-world.

2. Venus

We begin with an example of a problem where the formalization of the problem requires little physics and where the job is mainly to mathematize the situation.

2.1. Problem

'On the night sky, Venus can only be seen either in the morning or in the evening, because it is an inner planet relative to the Earth. What is the maximum number of hours that it can be seen during one night? The radii of the orbits of the Earth and Venus are 1 AE and 0.72 AE, respectively.'

2.2. Solution and analysis

First, we focus on the morning situation. In order to solve the problem, we need to determine the time it takes the Earth to rotate, so that an observer on Earth goes from seeing that Venus 'rises' to seeing that the Sun rises for an arbitrary configuration of the Earth, the Sun and Venus obeying the requirement that Venus is an inner planet. Then we need to determine the configuration that gives the largest value of this time.

The physics knowledge required to solve the problem is quite limited and restricted to the following facts: the Sun and Venus are approximately point-like objects as seen from the Earth, whereas the shape of the Earth is approximately spherical. Moreover, the orbits of the Earth and Venus are approximately circular and approximately in the same plane. We will assume that all these approximations are exact and that the rotational axis of the Earth is perpendicular to its orbital plane, even though it does in fact form an angle with the normal of this plane.

These basic physics facts and their idealization are already formulated in mathematical terms, so the part of the mathematization where the physical objects are represented in the mathematical universe has already been done from the outset. However, the question of the problem also needs to be mathematized; that is the next step. Mathematically speaking,

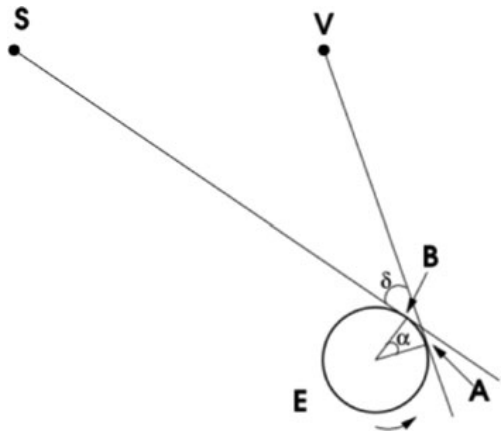

Figure 1. Sketch of the rising of the Sun and Venus as seen from the Earth.

the observer on the rotating Earth can see Venus, when she reaches the tangent to the Earth circle of the line originating in the point representing Venus. This is point A on Figure 1. The Sun rises when the observer reaches point B. Let α be the angle that the Earth rotates between the rising of Venus and the Sun. Then the time between the two risings of the astronomical objects is

$$T = \left(\frac{\alpha}{2\pi}\right) \cdot 24 \text{ hours}. \tag{1}$$

Hence, our next task is to find the maximum angle α for the allowed configurations. We now fix the position of the Earth and vary the positions of Venus. Due to the large distances between the planets and the Sun relative to the radius of the Earth we can represent the Earth as a point when we want to find this configuration (see Figure 2). Again, due to the large distances, the angle γ is equal to the angle δ in Figure 1.

The mathematical system encapsulated in Figures 1 and 2 marks the end of the formalization. In this process, the questions of when Venus can be seen and when the Sun rises

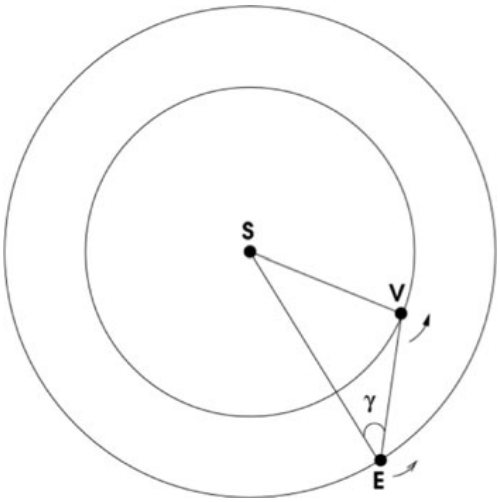

Figure 2. The configuration of the Earth, the Sun, and Venus.

are given mathematical formulations. When this mathematization is done, the problem is reduced to a mathematical problem of plane geometry, namely that of finding the maximum angle α for the allowed configurations. We start by noting that the angles δ and α are equal, so $\alpha = \delta = \gamma$. Hence, α attains its largest value when γ attains its largest value, which happens for the configuration where the lines from the Earth to Venus and from Venus to the Sun are perpendicular. Hence, we get

$$\sin\alpha = \sin\delta = \frac{R_{VS}}{R_{ES}}. \tag{2}$$

Here, R_{ES} is the distance between the Earth and the Sun and R_{VS} is the distance between Venus and the Sun. Thus,

$$\sin\alpha = \frac{0.72\ \text{AE}}{1\ \text{AE}} = 0.72 \Rightarrow \alpha = 0.80. \tag{3}$$

Hence, the time difference is $T = (\frac{0.80}{2\pi}) \cdot 24$ hours ≈ 3 hours. The evening situation can be solved in analogous manner.

This problem concerns the physical question of astronomical observations of physical planets. The solution process has a step that is mainly related to physics, namely some physical facts about the solar system and a step that is clearly within plane geometry, i.e. of a mathematical nature. In between these two steps, however, is the step of mathematization. Here, the physical ideas about the problem as well as the question are translated into mathematical language. In our experience with university students, this is the most demanding step for the problem solvers as they have to give a mathematical description of what it means that the Sun and Venus rise for an arbitrary configuration of the celestial bodies. In contrast, the physics ideas of the first step are rather limited and the mathematical analysis is not too difficult and students find these steps comparably easier to perform.

The above approach is that of a physicist with focus on getting a rough idea of what is going on by simplifying the situation as much as possible. Hence, a number of complications are neglected, including the non-circular nature of the orbits, that the planes of the orbits form an angle with each other, and that the rotational axis of the Earth forms an angle with the orbital plane.

3. Loose connection

Our second example is a problem where we have to make a thorough physical analysis before a mathematization can take place.

3.1. Problem

'A wall outlet, to which a water heater is connected, generates heat due to a loose connection in the outlet. What is the maximum heat generation that can occur in the outlet?'

3.2. Solution and analysis

In order to solve this problem, we need to make an analysis of what is going on in physics terms. A loose connection means that there is poor electrical contact; this can be represented as resistance to the electric current flow causing power to be dissipated as heat. Hence, we should represent the loose connection as a resistor. Likewise, the water heater can be represented by a resistor. Since the loose connection is positioned inside the outlet, it is reasonable to see this resistor as in series with the water heater resistor. This implies that the current through the two resistors is the same and that the voltage drop over both is the sum of the voltage drops over each element. This analysis of the situation based on physics knowledge has turned the problem into a formalized physics problem, namely to find the maximum power that is dissipated in the resistor representing the loose connection.

We denote the network voltage by V, the resistance of the water heater by R_h and it's dimensioned power by W_h, and the resistance and dissipated power of the loose connection by R_c and W_c, respectively. Finally, I is the current. The heat generated in a resistor R by a current I is

$$W = RI^2. \tag{4}$$

Now, we can mathematize the system: the voltage over the water heater is $V_h = IR_h$ and the voltage over the loose connection is $V_c = IR_c$. Since $V = V_c + V_h$, we get

$$V = I(R_h + R_c) \Rightarrow I = \frac{V}{R_h + R_c} \tag{5}$$

Hence, if we use Equation (4) we get that the power dissipated in the loose connection is

$$W_c = \frac{R_c V^2}{(R_h + R_c)^2} \tag{6}$$

After this mathematization of the system, we have now reduced our problem to the mathematical problem of finding the maximum of W_c with respect to the resistance of the loose connection R_c. We get

$$\frac{dW_c}{dR_c} = V^2 \frac{R_h + R_c - 2R_c}{(R_c + R_h)^3} = 0. \tag{7}$$

This happens when $R_c = R_h$, so the maximum heat generation in the outlet is given by

$$W_{c,\max} = \frac{R_h V^2}{(R_h + R_h)^2} = \frac{1}{4}\frac{V^2}{R_h} \tag{8}$$

Since $\frac{V^2}{R_h}$ is the dimensioned power delivery of the water heater, we get the result that the heat which can be generated in the outlet can be up to a fourth of that power.

This solution process contains two translations; first, we set-up a formalized physics problem based on a physical analysis and we realize that we have to find the maximum power that is dissipated in the resistor, when it is in series with a water heater of a dimensioned power. When solving this physics problem, we make yet another translation, but

this time it is a mathematization, where we set-up the equations for the electrical network. These equations follow immediately from the physical theory, i.e. the rules of electrical networks. Hence, this mathematization is interwoven with the physics of such networks. This leads to a pure mathematical problem, namely to find the maximum of W_c with respect to the resistance of the loose connection R_c.

In our experience, the challenging part for our university students is the initial physical analysis and in particular that the loose connection can be represented by a resistor in series with the water heater. When this idea is in place, the translation into a formalized physics problem and the mathematization of this problem is much less challenging to them.

4. Rod elongation

Our third example is a problem where it is required to perform a mathematization before physics can be brought to bear on the problem.

4.1. Problem

'A rod is hanging from one of its ends. By how much is it stretched under its own weight?'

4.2. Solution and analysis

The basic physical assumption is that the rod can be divided into small pieces that each is elongated by gravity. In order to apply the physics of elasticity to this situation, we need to make a mathematization and translate the problem into the domain of calculus by dividing the rod into small pieces (see Figure 3).

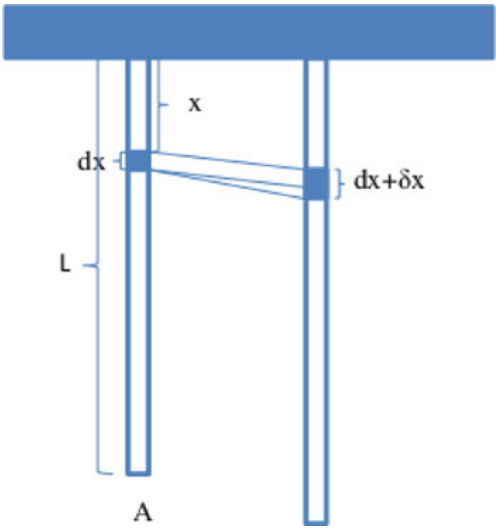

Figure 3. The elongation of a rod under its own weight.

We look at a rod piece of height dx positioned at x. This piece is elongated by δx. By adding up the elongation of all the pieces, we get the total elongation ΔL of the rod:

$$\Delta L = \int_0^L \delta x \, . \tag{9}$$

This is a mathematization of the situation based on a modelling of the rod with a rather limited content of physics.

Now we can bring physics to bear on the problem and turn it into a formalized problem: what is the gravitational pull on the piece dx at position x? And under the assumption of linear deformation, what is the total elongation of the rod? The physical domain of elasticity theory gives us that the force per area is proportional to the relative elongation:

$$\frac{F}{A} = Y \frac{\Delta l}{l} \tag{10}$$

Here, Y is Young's modulus. In our case Δl corresponds to δx, whereas l corresponds to dx. F is the gravitational pull experienced by the piece from below. If M is the mass of the entire rod, then $\frac{L-x}{L}M$ is the mass below x and the gravitational pull of this part of the rod is $\frac{L-x}{L}Mg$. Applying this to our situation, we get

$$\frac{\delta x}{dx} = \frac{1}{Y} \frac{L-x}{L} \frac{Mg}{A} \tag{11}$$

We can now make a mathematical analysis of this formalized physics problem; we find that the total elongation of the rod is given by

$$\Delta L = \frac{Mg}{YA} \int_0^L \left(1 - \frac{x}{L}\right) dx. \tag{12}$$

The solution is

$$\Delta L = \frac{1}{2} \frac{MgL}{YA} \tag{13}$$

In this solution process, we first employ some limited physics ideas in the beginning (that the parts of the rod would be elongated by gravity), then we do a mathematization and then we apply physics. Hence, the mathematization and involvement of physics are two distinct sub-processes with the mathematization process coming prior to the application of physics.

In our experience, the most demanding part of this solution process for our students is the mathematization based on the idea that the rod can be divided into infinitesimal pieces, each of which experiences an elongation, and hence that we should use infinitesimal elongations of infinitesimal pieces. Of course, the application of physics is also challenging to some students.

5. Atmosphere

Our final example is a problem where the application of physics and the mathematization are so interwoven that it is difficult to separate the two processes from each other.

5.1. Problem

'If we assume that the temperature of the atmosphere is constant, how does the pressure change with height?'

5.2. Solution and analysis

Setting up the system consists in seeing the atmosphere as a vertical column that can be 'cut' into horizontal, thin slices. We look at a horizontal layer of area A and thickness dh at a height h. The vertical force per area from below is $P(h)$, where $P(h)$ is the pressure at height h and from above is $P(h + dh)$. The incremental pressure is given by

$$dP = P(h + dh) - P(h). \tag{14}$$

We assume that the atmosphere is stationary; hence, the force from below must exceed the force from above by the weight of gas in the section between h and $h + dh$. The force of gravity per unit area from the gas molecules in the section is $g\rho dh$, where ρ is the density. We get

$$dP = -g\rho dh. \tag{15}$$

The density is height dependent, so we need to rewrite ρ. We can do so by using the ideal gas law,

$$\rho = m\frac{N}{V} = m\frac{P}{k_b T} \tag{16}$$

where N is the number of air molecules in the volume and m is the mass of each molecule. Hence, we get the differential equation

$$\frac{dP}{dh} = -gm\frac{P}{k_b T} \tag{17}$$

We know that the pressure at the surface of the Earth is 1 atm. A mathematical analysis of Equation (17) gives us the solution:

$$P(h) = Ce^{-gmh/k_b T}. \tag{18}$$

We get the initial condition that $P(0) = 1$ atm, so $C = 1$ atm. Hence, we obtain the final result

$$P(h) = 1\text{ atm} \cdot e^{-gmh/k_b T}. \tag{19}$$

Quite similar to the rod elongation problem, we focus on horizontal, thin slices and we also have to make a mathematization using an infinitesimal height *dh*. Now we make a physicalization, where we first need to analyse the forces acting on the section (Equation (14)), which corresponds to Equations (9) and (11) in the 'rod elongation' problem. However, for the present problem we also need to apply the ideal gas law to eliminate the density ρ, which is height dependent, from the resulting equation (Equation (15)). Hence, in order to arrive at the mathematical problem, i.e. the differential equation in Equation (17), we need to mix physics and mathematics in an intricate way, e.g. using physics knowledge to realize that ρ is non-constant and that it can be eliminated by using the ideal gas law. Hence, compared to the 'rod elongation' problem the physical analysis and the mathematization sub-processes are harder to disentangle. Moreover, the mathematization process is not in the same way as the mathematization process in the solution to the 'loose connection' problem a straight forward setting up of the mathematical equations from the physical theory, since it requires an adaptation of the equation to the present situation. In our experience, many students solving this problem experience difficulties with this entangled mathematization.

6. Discussion: four scenarios of problem solving in physics

The above analyses of four solution scenarios can be used to inform a discussion of how physics problem solving can assist the development of mathematization competence. The analyses show four different relationships between mathematics and physics in the solution processes. We can summarize the findings with a didactical model for problem solving consisting of three stages and two processes.

The crucial bits, or steps, of the problem solving are of course the two processes involved, and in particular the first one of these, i.e. formalizing the problem into a *formalized problem*, which is a problem formulated in physical and/or mathematical terms, drawing on the usual notation within these two fields. The *formalization* may be seen to consist of two subprocesses that may be interwoven: a *systematization* and a *translation*. In the systematization, one must specify the interesting and central bits in the situation one wishes to capture. This involves deciding on the relevant factors – and disregarding those considered irrelevant for the given question. The translation consists in expressing the problem in physical and mathematical terms by deciding on variables, naming these, writing up expressions to be considered, equations to be solved, etc. Now, depending on the problem in question, this translation may be either a *mathematization* or a *physicalization*, depending on whether the problem is translated into a mathematical domain or a physical domain or a combination of the two. The formalized problem can be *solved*, eventually leading to a *result*, not only of the formalized problem, but also of the initial unformalized problem.

Mathematization and physicalization are processes which to some extent are described in the mathematics and science education literature. So, allow us to briefly recapture what we know about these. *Mathematization* is the process of taking, or mapping, the problem formulation for an extra-mathematical domain to mathematical objects and relations between them in some mathematical domain. Within mathematics education literature, the process of mathematization is well-described as one of the phases of performing mathematical modelling, i.e. one of the phases in the so-called modelling cycle (e.g. [2,3]). Several empirical studies show that mathematization is particularly difficult for students (e.g.

[4]), and that knowing a lot of different mathematics does not necessarily enable one to perform mathematization.([5]) Furthermore, the choice of the mathematical domain (e.g. plane geometry or differential calculus) and mathematical entities (e.g. right-angled triangles or differential equations) presupposes that the problem solver already has an idea of potential mathematical domains to choose, and how to use the mathematical representations to obtain the needed answers. This is referred to as part of *implemented anticipation*,[3] and is considered the element of mathematization, which is perhaps the most difficult for students.[6] As an example, if one realizes that a mathematization will lead to a system of non-linear differential equations, one may consider whether a different systematization (and hence a different mathematization) would be possible that lead to an easier mathematical problem-solving process.

As for *physicalization*, it is a process analogous to the mathematization process: a question posed in a real-world context, and typically formulated in everyday language, is taken into the domain of physics. The word physicalization is not in widespread use in the physics education literature, although the idea is not completely unknown. For example, Heller and Hollabaugh [7] hints at what we mean by physicalization when they describe the decisions required for solving a real-world problem: the problem solver needs to decide: (1) what variable(s) are relevant to answer the question; (2) which physics concepts and principles would be useful to determine that or those variables; (3) what information is required; and (4) how that information can be found. The literature abounds with research findings showing that students do not make such physicalizations when they solve traditional textbook problems, but rather follow less scientific strategies, such as the plug and chug strategy (see, e.g. [8]).

When the mathematization and/or the physicalization are finalized, the problem solver is faced with solving a formalized problem. Whether this process is a process that involves mainly mathematics or physics of course depends on the given problem (and on its systematization and translation). For mathematics this process consists in answering the mathematically translated question by means of mathematical methods and techniques within the chosen mathematical domain. For physics, the process involves answering the physical questions by means of physics concepts and theories of the physical domain of the question. But often, if not always, it also requires applying mathematical methods and techniques.

Solving the formalized problem, in mathematics as well as in physics, is well-described in the educational literature (see, e.g. [8–10]), and is as such not the main issue of this article. Hence, we shall not go into depth with this. We would like to remark, however, that in our four examples the mathematization is part of the formalization process of the problem rather than the solution of the formalized problem. Thus, in the case of these examples, the formalized versions of the problems do not train the mathematization competence.

The four scenarios illustrate that the interplay between mathematization and physicalization can assume very different forms. In the first example of 'Venus' little knowledge from physics is needed, and the solution of the problem mainly consists in performing a mathematization of the planetary system leading to a purely mathematical problem solving. The second example of the 'loose connection' requires a physical analysis that leads to a use of a physical theory (on electrical networks) and the equations follow immediately from this. In this case, the physicalization comes prior to the mathematization, which, once performed, reduces the rest of the problem to mathematical problem solving. In the third example of 'rod elongation' an initial, but crucial, mathematization has to be

made prior to physics coming into play; however, the way to perform the mathematization is informed by the physics. And finally, in the fourth example of the 'atmosphere' the mathematical and physical problem formulation sub-processes are so interwoven that it becomes meaningless to try to disentangle them.

The scenarios also illustrate that it can be difficult to separate the physicalization and mathematization parts of the solution process. The distinction between mathematics and physics is not easy to draw. Is the assumption that the planets in the first example move in circular orbits physics or mathematics? Is the division of the rod and the atmosphere into infinitesimal pieces in the third and fourth example, respectively, physics or mathematics?

In the science education literature, Karam and Pietrocola,[11] and Karam, Pospiech, and Pietrocola [12] distinguish between technical skills and structural skills, when using mathematics in physics:

> The technical skills are related to the instrumental ability to apply mathematical rules and algorithms in physics, whereas the structural skills are associated to the use of mathematics to structure physical thought and to the recognition of the deep connection between the physical content and the mathematical formulation of a particular concept.[12 , p.2]

In this respect, the instrumental ability in relation to technical skills may be considered part of the mathematical problem solving, whereas the structural skills are part of the problem formalization and, hence, mathematization. As for the structural skills, in particular those related to the use of mathematics to structure physical thought, these are required in all the four scenarios.

The four problem-solving scenarios potentially also call for implemented anticipation (cf. earlier [3]). In the problem on 'Venus' the problem solver must anticipate that once in the domain of plane geometry, she will be able to solve the problem. In the problem on the 'loose connection' the problem solver must anticipate that the mathematization will lead to a manageable calculus problem of finding a maximum. In the problem on 'rod elongation' the problem solver must anticipate that infinitesimal calculus will lead to a formulation of a meaningful expression, and further that elasticity theory indeed will make it possible to solve the problem. Finally, in the problem on the 'atmosphere' the problem solver must again anticipate what to do with a mathematized expression in the domain of infinitesimal calculus; in this case leading to a differential equation that may be solved analytically.

7. Conclusive remarks and potential remedies

From a societal point of view, the non-teaching of mathematization is of course rather inexpedient, not least because society believes that students are able to solve practical real-world problems, i.e. unformalized problems. But to do so, as illustrated above, requires the possession of a *competence.* And as with all other competences, these do not emerge out of nowhere; they must be developed and trained. The expectations of both society and students are to a high degree directed towards the competence of mathematization. The students must be able to solve problems by means of mathematics. But how may the educational system teach them this?

Above, we have claimed that physics teaching, if it focuses on solving unformalized problems, has the potential of training this competence and more generally a *formalization competence*. This claim is supported by examples of solutions to unformalized problems and is

Figure 4. A didactical model of problem solving consisting of three stages (boxes) and two processes (arrows).

encapsulated in the model of the problem-solving process in Figure 4. We believe that this model captures the important steps in the solutions of the unformalized problems, and we have shown this to be the case for the four unformalized problems described above. Moreover, it is our experience with teaching problem solving in physics based on unformalized problems that the formalization step of the problem solving is very demanding on the part of the students no matter what kind of formalization is required. In some cases, it is the physicalization that causes difficulties. However, often the students are capable of describing in qualitative physics terms what is going on, but are incapable of performing the needed mathematization. Hence, we claim that unformalized physics problems often can be used as a training ground for mathematization, thus, realizing the potential of physics in this respect.

Still, if physics teaching is indeed to realize the potential of training mathematization, the physics teachers have to be made conscious that one of the main purposes of their teaching is to develop and train mathematization. They must come to regard physics teaching as a terrain for practicing mathematization, i.e. seeing physics as a means and mathematization as an end, not the other way round. More utopian, one might also imagine that the teaching of mathematics and physics are joined into one subject of mathematization.

As mentioned already, the above considerations are based on our experiences with teaching physics at undergraduate level. However, physics at earlier educational levels might not be up for the task of training mathematization. The subject of physics changes character from being theoretically oriented at university level to being a phenomenon and experimentally oriented subject at the earlier educational levels. On the other hand, in some cases such as Denmark, mathematics teaching at the primary and lower secondary level is more focused on training competences, including mathematization, and hence in this case mathematics teaching actually functions as a training ground for mathematization. At upper secondary level in Denmark, however, mathematics is often taught more traditionally; hence, physics at this level can potentially become the training ground for mathematization.

The model in Figure 4 seems to be general enough that it is also applicable to the problem-solving process in other subjects that apply mathematics. However, the formalization step is different for different subjects. In physics, a crucial stage in the problem-solving process is to see and formulate the problem in terms of physics, so that the theoretical machinery of physics can be brought to bear on the problem. However, other approaches to problem solving do not draw on a theoretical corpus. Some of these approaches involve common sense arguments rather than theoretical considerations. This is the case for much of the mathematical modelling tasks reported in the mathematics education literature. Other approaches draw on a tradition and a canonical way of attacking problems, but in a way so that it is not theoretically grounded. As we have argued above, for problem solving in physics, mathematics and the theories of physics are closely intertwined and the

mathematization often happens intertwined with the physicalization. Hence, for physics, in contrast to some of the other formalization approaches, in particular common sense arguments, the mathematization is done in an interaction with extensive subject knowledge. This subject knowledge needs to be acquired in some way or the other. Consequently, from the mathematics point of view the training of mathematization using physics comes with a price, namely that it needs to be trained on a physics syllabus, and this typically requires that the students learn some physics in the process. At the same time, the students are trained in a physicist approach to problem solving, namely to focus on the essence of the situation and leave the rest aside – or in the (perhaps apocryphal) words of Einstein, 'Everything should be made as simple as possible, but not simpler.' It is up to the educational system(s) whether this trade-off between training ground for mathematization and physics learning is acceptable. But, at least at the upper secondary level and beyond, physics has a crucial role to play as a training ground for mathematization.

Notes

1. As said, the situation described here mainly concerns the mathematical program of the classical upper secondary school, the so-called 'STX', and partly also the business upper secondary school, called 'HHX'. In the technical upper secondary school, called 'HTX', there is a more direct focus on mathematical modelling, and hence also on mathematization.
2. This course was developed by the first author and has been taught since 1975.
3. The problems are taken from the collection of problems in [1]. The collection can be accessed at http://milne.ruc.dk/ImfufaTekster as item number 474.

Disclosure statement

No potential conflict of interest was reported by the authors.

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I den følgende kommentar i MONA, dansk matematik- og naturfagsdidaktisk tidsskrift for undervisere, forskere og formidlere, var der på redaktionens opfordring tilføjet en uddybende sætning, som desværre faldt ud i den endelige trykning. Efter "projektforløbet." side 91 linje 9 fra neden skulle der have været indføjet: På samme måde, som man bredt i samfundet tydeligt kan oplever, at stillingtagen til astrologi, alternative behandlinger m.m. mere afhænger af folks større eller mindre naturvidenskabelige forankringer end af modtaget videnskabsteori undervisning.

Kompetencemålsstyring og competitis

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Kommentar til Jette Reuss Schmidt: "Neoliberal styring af dansk naturfagsundervisning siden årtusindskiftet – fra pensumitis til competitis?", MONA, 2017-1.

Jette Reuss Schmidts (JRS) artikel i sidste nummer af MONA var befriende læsning. Den placerer en didaktisk diskussion i dens politiske kontekst. Derved er artiklen, for mig at se, mere realistisk end så megen anden didaktisk litteratur der byder sig til med positive overvejelser afkoblet fra de organisatoriske og politiske sammenhænge overvejelserne udspringer af og vil blive brugt i. Forestillingen, først positiv evidens baseret på uafhængig didaktisk forskning, dernæst politisk beslutning på et oplyst grundlag, er i bedste fald naiv, i værste fald skræmmende totalitær. Didaktik leverer ikke positiv evidens som medicin kan gøre det. Jeg kender ikke til et eksempel på positiv didaktisk evidens på linje med fx evidensen af virkningen af HPV-vaccinen. Hvis didaktikere udgiver sig for at kunne levere positiv evidens, risikerer de at politisere som politologer gør det når de legitimerer "den nødvendige politik".

JRS afdækker i artiklen hvordan en, blandt andet OECD-inspireret, neoliberal styring har præget rammesætningen af dansk naturfagsundervisning siden årtusindskiftet. I forlængelse af den overordnede New Public Management (NPM) top-down-styring er et af styringsmidlerne et omfattende sæt kompetencemål. Derfor spørger JRS om ikke naturfagsundervisningen er skiftet fra at lide af sygdommen pensumitis til (også) at lide af sygdommen competitis. JRS har her konstrueret ordet competitis som en sammensætning af de engelske ord competence og competition. Degenereringen af kompetencemål til en lang række løsrevne "faglige" byggesten skal ud over et skift fra pensumorientering til kompetenceorientering (competence) også, som et selvstændigt formål, sikre at kompetencemålene leverer gode vilkår for konkurrence og benchmarking (competition).

Ordet pensumitis stammer fra en artikel som jeg skrev i 1995 (Jensen, 1995). Synes jeg da stadig at pensumitis er et hovedproblem, eller er det blevet competitis der nu

er udfordringen? Og i bekræftende fald: Hvad kan naturfagsdidaktik bidrage med for at begrænse ulykken?

Som citeret af JRS beskrev jeg sygdommen pensumitis således:

> Den består i, at pensum forveksles med faglighed (indsigt, forståelse og kunnen) og i, at undervisning forveksles med tilegnelse med den konsekvens, at tilrettelæggelse af undervisning indskrænkes til fastlæggelse af pensum. Tænk fx på: Højt fagligt niveau som bilist = stort og rigtigt sammensat pensum til teoriprøven. (Jensen, 1995).

Pensumitis har overfladelæring i stedet for dybdelæring som konsekvens. Den opfattelse fik tilslutning i det daværende undervisningsministerie i en grad så ordet blev brugt i offentligheden af to på hinanden følgende undervisningsministre. Men hvordan bekæmpe sygdommen?

I 1999 gav Mogens Niss med artiklen "Kompetencer og uddannelsesbeskrivelse" (Niss, 1999) et bud. Brugen af ordet kompetence var her hverken neoliberalt eller OECD-inspireret. Og heller ikke direkte arbejdsfunktionsorienteret. Ordet var valgt som en nøgle til mere grundlæggende beskrivelser end pensumbeskrivelser af fag, i første omgang faget matematik. I kølvandet fulgte den omfattende KOM-rapport (Niss & Jensen, 2002) vedrørende matematik og inspireret heraf en tilsvarende rapport vedrørende naturfag (Dolin, Krogh & Troelsen, 2003). Der fandt altså en markedsføring af et fagrettet kompetencebegreb sted i samarbejde imellem centraladministrationen og de matematikdidaktikske og naturfagsdidaktiske miljøer. Man kan derfor spørge: Er didaktikmiljøerne ansvarlige for competitis-tilstanden nu om dage som JRS beskriver den?

JRS skriver om den senere udvikling:

> Dermed flyttes magten fra mødet mellem underviseren, den studerende og det faglige indhold til en output-fikseret, markedsrelateret central målsætning hvor markedet personificeres med den kommende arbejdsgiver.

I modsætning hertil bedømmer JRS didaktikmiljøernes kompetencerapporter som orienteret imod almendannelse og bottom-up-processer. JRS giver altså ikke didaktikmiljøerne ansvaret for udviklingen imod competitis. Jeg kan tilslutte mig frifindelsen med følgende citat fra afslutningen af Mogens Niss' første artikel om kompetencer:

> *Advarsel mod behaviorismefælden. Det skal ikke skjules, at bestræbelserne på at levere en kompetencebaseret karakterisering af et undervisningsfag, som her matematik, også kan rumme visse farer for misbrug. Den vigtigste fare består i at forlade det sammenfattende og overordnede beskrivelsesniveau for i stedet at gå i detaljer med udparcellering og selvstæn-*

> *diggørelse af kompetenceelementer, et forehavende der måske kunne friste nogle, i disse tider, der er så besat af afrapportering-, dokumentations- og evalueringstiltag.* (Niss, 1999)

Kan Mogens Niss (m.fl.), på trods af hans således forudseende advarsel, anklages for at have gjort kompetencetænkning i almindelighed til en medvind som førte competitis-sejladsen med sig?

I 1996 skrev jeg en artikel med overskriften "Kan "konstruktivismen som ide" udrydde pensumitis?" (Jensen, 1996) som reaktion imod at pædagoger og didaktikere på det tidspunkt havde set lyset i "den konstruktivistiske ide". Ikke fordi jeg var imod den (selvfølgelige) konstruktivistiske ide. Men hvor langt rækker ideer egentlig ved sygdomsbekæmpelse? I konklusionen på artiklen skrev jeg:

> Især er det vanskeligt at tro på, at "konstruktivismen som ide" kan udrydde pensumitis, hvis det indebærer, at ideen skulle overflødiggøre styringsbehovene i uddannelsessystemet. Afvejningen af styringsmålsætninger over for styringsmidler og deres skadevirkninger er en sag af uddannelsespolitisk nærmere end snævrere pædagogisk karakter. (Jensen, 1996).

Nu mere end tyve år efter kan vi erstatte "konstruktivismen som ide" med "uddannelsesbeskrivelser ved kompetencemål" i citatet. Det er, som sagt indledningsvis, også i dag naivt at forestille sig didaktik og pædagogik løsrevet fra uddannelsespolitik. Og skal man tænke positivt

> over, hvordan man med ord bedst bidrager til at fremme den gode undervisning, er det bedre på sokratisk vis at markedsføre et skældsord som dåseåbner for problembevidsthed end at påfylde omgivelserne positiv "sandhed" gennem plusord. (Jensen, 1996).

Skældsordet pensumitis er således stadig et godt ord som ikke så let kan drejes væk fra dets intenderede formål som plusordet kompetencer kan. Samtidig er det mere realistisk at betjene sig af skældsord end plusord. Fagdidaktik er først og fremmest leveringsdygtig i konklusioner om hvad der *ikke* virker, frem for konklusioner om hvad der virker.

Jeg er altså glad for JRS' markedsføring af skældsordet competitis. Og det tror jeg også at Mogens Niss er.

Alligevel synes jeg ikke skældsordet competitis overflødiggør plusordet kompetencemålstyring som positiv replik til pensumitis. Der er brug for kompetencetænkningen ved bottom-up-udvikling af undervisning. Et eksempel jeg har inde på livet, er fx min søns (Tomas Højgaards) bidrag til udviklingen af lærebogssystemet Matematrix til folkeskolens matematikundervisning (Matematrix, 2001-2008). Der er ikke her tale

om et top-down-bestillingsarbejde. Men derimod om en bottom-up-proces der så vidt jeg kan vurdere, er positivt inspireret af arbejdet med den omtalte KOM-rapport.

JSR skriver i afslutningen af sin artikel:

> Som naturfagsundervisere og naturfagsinteressenter i øvrigt må vi til stadighed have fokus på hvilke unge mennesker vi ønsker at skabe, og dermed på hvilket samfund vi ønsker

og

> ...argumentere for en forståelsesorienteret naturfagsundervisning hvor man netop ikke forsøger hurtigst muligt at komme fra input til output, og hvor formålet med naturfagsundervisningen står i centrum.

Til fokusering af både forståelser og formål kan kompetencetænkning efter min vurdering være værdifuld. Med forbehold for meget lille erfaring med folkeskolens undervisning i naturfag forestiller jeg mig at følgende kompetencetænkte fokusering kan være af (positiv) værdi: Ved siden af viden om naturforhold skal folkeskolens naturfagsundervisning lægge kølen til elevernes *kompetence* i at vurdere *naturvidenskabelig evidens* som kernen i deres *naturvidenskabelige dannelse*.

Fra min universitetsundervisning ved jeg at det er en svær og stor udfordring som må basere sig på en fagkultur i sin helhed snarere end overleverbare opskrifter. Som eksempel vil jeg nævne min erfaring fra vejledning af et projekt på andet år af den naturvidenskabelige basisuddannelse på RUC. Projektet fyldte halvdelen af de studerendes tid i et semester. De studerende havde i udgangspunktet ønsket at arbejde med jordstråler. Siden fik jeg overtalt dem til at dreje projektet imod en komparativ modstilling af jordstråler og radioaktive stråler. Man kan hverken tage, se eller føle på hverken jordstråler eller radioaktive stråler. Hvordan kan vi fysikere da vide at der er evidens for det ene og ikke for det andet? Er der ikke blot tale om en fysikerfordom? Eksemplet viste mig at det der rykkede på de studerendes forståelse af hvad der er evidens for, og hvad ikke, ikke så meget var argumenter af bestemte slags som det var deres integrerede erfaringer gennem hele projektforløbet. Kort sagt: Evidens er ikke så klar og entydig en størrelse. Den skal erfares i dens mange versioner.

Ved reklame for naturvidenskab og popularisering af naturvidenskab er det næsten udelukkende naturvidenskabelige resultater, og næsten aldrig naturvidenskabelige metoder, der er i centrum for opmærksomheden. I forhold til "hvilke mennesker vi ønsker at skabe, og dermed hvilket samfund vi ønsker", er det imidlertid vigtigt at åbne op for en orientering imod metoderne og vurderinger af naturvidenskabelig evidens. Og det kræver et skift fra pensumtænkning til kompetencetænkning.

Men kompetenceordet alene flytter selvfølgelig ikke noget. Dybest set drejer det sig om at bevare og udvikle metodisk orientering og forståelse i og omkring den naturfaglige kultur i folkeskolen. På trods af, ikke i kraft af, top-down-styringen. Uanset neoliberalisme eller ej kan toppen kun stå i vejen så længe den ikke erkender at dens mangel på brede naturvidenskabelige erfaringer nødvendiggør udpræget bottom-up-mobilisering.

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Den følgende artikel har været trykt som kapitel 7 i T. Sibbald (Eds.), *Teaching Interdisciplinary Mathematics*. Champaign, IL: Common Ground Publishing. Den var længe under vejs. Filosofiske essays er ikke en alment accepteret forskningsgenre inden for matematik- og naturvidenskabsdidaktik.

CHAPTER 7

Disciplines and Ways of Perception: Linking Interdisciplinarity and Competences

Jens Højgaard Jensen and Uffe Thomas Jankvist (Denmark)

INTRODUCTION

Since the 1960s, at least, the word "interdisciplinarity"[15] and, more recently, the word "competence" [16] have become *buzzwords* in both science and mathematics education discussions round the world. Only seldom, however, are the discussions of interdisciplinarity and competence related to each other. Usually these discussions run on separate tracks within very different discourses in the science and mathematics education communities.

In this chapter, we aim to argue and show that the two discussions are in fact related. More precisely, the manner in which interdisciplinarity comes into play – or is at all possible – in given activities depends on the degree to which the involved academic disciplines are borne by certain competences as "way of perception." In order to argue our case we first illustrate that disciplines are in fact very different, and hence also disciplinarity. Next we introduce the notion of way of perception by means of an authentic example. We then argue for the existence of three different kinds of interdisciplinarity, each driven by a specific demand. This leads to a discussion of competences in relation to way of perception, and in particular three different types of competences the mixture of which is defining for what constitutes a discipline, and hence also the conditions for interdisciplinary collaboration between disciplines, which we discuss in the section on linking interdisciplinarity and competences. At the end of the chapter we relate our discussion of interdisciplinarity and competences to potential shortcomings of Jantsch's taxonomy from 1972.

It should be noted that throughout the chapter we shall mainly relate our discussion to university level. The reason for this is that the objects of our

[15] A few examples are: Beckman 2009; Boisot 1972; Browne 2002; Czerniak 2007; Develaki 2008; Haynes 2002; Hurley 2001; Jensen 1991; Karlqvist 1999; Klein 1996; Love 2013.
[16] A few examples are: Blomhøj and Jensen 2007; Højgaard 2012; Jankvist and Kjeldsen 2011; Michelsen 2005; Niss 2003; Niss and Højgaard 2011; Reiss 2013.

consideration are most visible at tertiary level, also in terms of examples. But the discussion is relevant for lower educational levels as well and in particular for upper secondary level (see e.g. Jankvist, Nielsen and Michelsen 2013).

Characterizing Disciplines

Disciplines—and hence also disciplinarity—may be characterized or distinguished from one another in many different ways. Liedman (1972), for instance, mentions three: analyzing a discipline according to its origin; according to its content; or according to its function. In the following we shall provide a characterization according to the different kinds of origin (genesis) of disciplines.[17] This characterization shall suffice to illustrate the point that disciplines are indeed different in the way they are organized and constituted.

Discipline originating from knowledge and literacy, for example the disciplines of history or mathematics, which have existed as domains of human knowledge long before the existence of historians or mathematicians. Of course, nowadays it may also be a profession to be a historian or a mathematician. The genesis was of course aspects of human knowledge and literacy existing in advance, but needing administering and further development.

Discipline originating from a profession, for example a chef, a medical doctor, or a carpenter. The genesis here is differently defined areas of human activity, which have been specialized through the societal division of labor to be governed by professionals. This has then resulted in a need for professional training and eventually, for example, research of nutrition, medical science, and research into the construction of buildings.

Discipline in the form of a school subject, for example history, mathematics, home economics, shop class, or English. In the school's organization of disciplines (subjects) the demarcation takes place both in relation to areas of knowledge and literacy, such as in the cases of history and mathematics, and in relation to areas of activity, such as in the cases of home economics and shop class. But the school also constructs subjects of its own, as for example classical civilization or the piecing together of cultural fragments, literature fragments, and language fragments to a subject such as English. And since one of the responsibilities of the universities is to educate school teachers, the arrangement of the school has influenced the subject/discipline division at the universities.

Discipline in relation to a problem area, for example environmental studies, studies of internationalization, or studies of cultural encounters. Also, the universities develop new disciplines due to societal problem areas. Over

[17] This characterization has only previously been presented in Danish (Jensen 1995).

time this may lead to new professions, as for example in the case of computer science.

In summary, the above division is to do with the origin (genesis) of the disciplines. Was the discipline an area of knowledge and literacy prior to becoming a profession being a carrier of such knowledge and literacy? Or was the discipline originally an area of activity, which developed an associated knowledge and literacy? In all circumstances, a discipline changes and evolves over time, including its interrelations between the aspects of knowledge and activity of a given discipline.

As mentioned, our distinction between disciplines according to their different kinds of origin is but one out of many possible. Alternatively, one might have attempted to map disciplines' various functions in society, according to their content, [18] or as being governed by one or several paradigms, as being of a more or less constructive or analytical character, a more or less empirical or theoretical character. But for our purpose the above characterization shall suffice to illustrate the crucial message that no matter how you distinguish disciplines from one another, sociologically or epistemologically, they are very diverse. For that reason, interdisciplinarity is also a very different and diverse thing dependent on the kinds of disciplines that are to interplay. Hence, it is a difficult task to generalize about interdisciplinarity.

Disciplines as Ways of Perception

One potential way to view disciplines is through their dependency on an associated way of perception. All disciplines of course have an associated way of perception as part what constitutes the discipline. Yet for some disciplines this is more decisive than for others. The main message of this very chapter is to show that interdisciplinarity in relation to given disciplines will depend on the weight of their associated way of perception as part of what constitute these disciplines.

In this section we illustrate what we mean by way of perception through an example involving two disciplines, each heavily characterized by their associated way of perception; geology and physics.

[18] For an example of a categorization according to content, although also in Danish, see Jensen (2012).

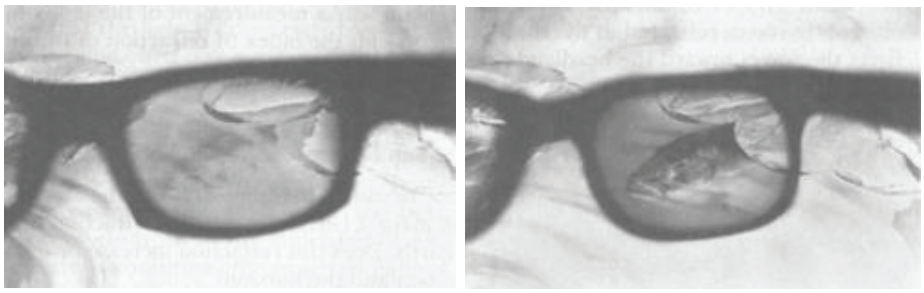

Figure 1: To see is to disregard. On the left, seen through ordinary sunglasses the goldfish is practically invisible due to the light's reflection off the water surface. On the right, the goldfish is seen clearly through the Polaroid sunglasses, because the polarization sorts out, or disregards, the obscuring part of the light.
Source: Pictures taken from Ohanian, 1989, 915

W*ay of perception* may be illustrated by means of figure 1. Having a certain way of perception is similar to viewing the world through a particular Polaroid filter. The filters of the geologists and the physicists are different. Metaphorically speaking, they are trained in applying different directions of polarization. But geologists and physicists have in common that their disciplines are closely connected to specific filters. Geologists do not see mountain formations as the rest of us. Mountains and their surrounding landscapes, function for them as original sources do for historians. Geologists are trained as *natural historians*. In a similar manner, physicists do not see motion as non-physicists do. They see them as manifestations of all-embracing mathematical patterns of motion. As for philosophers, single phenomena serve as take-off for generalizations. Physicists are trained as "mathematical thinkers of essence." It is not necessarily the case that geologists miss that mountains may be perceived as something else than historical remnants. Nor is it necessarily the case that physicists miss that motion may be perceived as something else than a mathematical pattern. What they do is that they disregard, and *disregarding* is not the same as *missing*. By disregarding, for them obscuring phenomena and circumstances, they are each capable of seeing "goldfish" in their own way – as when in figure 1 seeing through the Polaroid glasses.

Allow us to illustrate our point with an anecdote. The father of the first author of this chapter was a professor of physics at the University of Copenhagen in the period from 1960 to 1988. During this period he was the chairman of a committee looking into the need for Danish expertise in exploring for oil from the point of educating such experts at the university. Since exploring for oil involves knowledge of both the subsoil and knowledge of methods for examining the subsoil, the committee consisted of geologists and physicists. The geologists and the physicists of the committee were convinced of the necessity of the expertise of one another. But what it came down to was that the geologists acknowledged physics only as a useful tool

for certain practical purposes, whereas geology was an acquired way of perception a la Figure 1, making it possible for the qualified eye to spot circumstances which remain hidden for the non-qualified.[19] With the physicists, the exact opposite was of course the case. For that reason, the geologists found that a practically normal geology program supplied with a minor course in the methods of interpreting seismic data would make up the best education of experts in oil exploration. On the other hand, the physicists found that a more or less normal program in physics supplied with a minor course in becoming familiar with the order of the different subsoil layers would make up the best education of experts in oil exploration. Furthermore, it was the impression that this situation was not unique for the University of Copenhagen. The oil industry was suffering from this worldwide.

The case of geology and physics is not the only example. The relationship between mathematics and physics may be described in a similar manner, where mathematicians only see physics as one out of a long line of exemplary applications of mathematics, and physicists often regard mathematics as no more than a useful toolbox of concepts and results (e.g. see Jensen, Niss and Jankvist 2016). Normally, professionals are aware of the fact that they disregard things when putting on their Polaroid glasses. But typically, they miss the point that glasses with different directions of polarization than their own exist. Other disciplines than one's own are perceived as collections of concrete facts, skills, and recipes more than alternate ways of perception. Usually, one has never experienced seeing the world through others' glasses.

Three Kinds of Interdisciplinarity

As previously argued (by the example of genesis), disciplines are diverse. Interdisciplinarity, besides being dependent on the kinds of disciplines that are to interplay, is also dependent on what the *purpose* for interdisciplinarity is. It is impossible to practice interdisciplinarity, if one does not realize *why* interdisciplinarity is needed in the first place.

As disciplines become more specialized, the difficulty in identifying oneself with the aim of other disciplines also increases. This has downsides, as in the example from oil industry when geologists and physicists find it still harder to communicate in an esoteric manner. The downsides of the increase in specialization of disciplines are therefore one of the reasons for the demand of interdisciplinarity: interdisciplinarity becomes more important to help build bridges between disciplinary cultures. For example, in the case of the experts exploring for oil, it is not primarily an education in oil exploration which is needed. The oil companies will deal with that. Rather what is in demand is

[19] Freely recited from the memory of Professor Henning Højgaard Jensen.

educating students and teachers to be able to think and perceive as a geologist and a physicist at the same time. However, the universities' organization according to specializations is a hindrance for meeting this demand. Increasing specialization in research and in the graduate programs since World War II has led to loss in overview across disciplines and made cooperation between disciplines more difficult. The word interdisciplinarity is marketed as an answer to these problems.

A completely different kind of demand for interdisciplinarity has to do with the problem and application orientation of research and education. Examples are didactics of science and mathematics as well as many of the social sciences. It is not only a matter of establishing dialogue between different disciplinary cultures, but more about the breaking down of disciplinary borders that are in the way of problems being addressed seriously – or objectively and reasoned. It is the disciplines that must be subordinated to the matter being addressed, meaning that interdisciplinarity is not an aim in itself, but a consequence of orientation towards problems and applications.

The concept of interdisciplinarity is used with very different meanings and for very different reasons. Besides the above mentioned two situations related to different kinds of demands, the concept is also used when creating a new discipline on the border of two existing disciplines. Examples are the various hybrids of biology and other disciplines, such as biochemistry, biophysics, bioinformatics, and biomathematics.[20] In summary, we thus identify three distinct uses of the concept interdisciplinarity according to demand:

Interdisciplinarity as integration of disciplines on the border of two existing disciplines. Examples are: mathematical economics between mathematics and economics; bioinformatics between biology and computer science; and geophysics between geology and physics. That is to say *interdisciplinarity is driven by a demand for specialization*: the demand for interdisciplinarity has the purpose of establishing new specialized disciplines with a focus in between already existing and well-established disciplines.

Interdisciplinarity as the integration of disciplines focusing on a certain practical area of problem and function. Examples are: business studies, which integrates economics, sociology, law, etc.; medical science, which integrates chemistry, physiology, psychology, etc.; engineering, which integrates mathematics, physics, geology, etc. That is to say *interdisciplinarity is driven by a demand for integrating elements from basic disciplines to an applied discipline*: the demand for interdisciplinarity has to do with promoting applicability of basic disciplines in relation to practical problem solving.

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[20] In the taxonomy of Jantsch (1972), such hybrids are referred to as examples of transdisciplinarity (see epilogue of the chapter).

Interdisciplinarity as understanding and overview across disciplines. Examples are university studies with a double major, i.e. studying two very different disciplines on an equal footing.[21] That is to say *interdisciplinarity is driven by demands for disciplinary cosmopolitans*: the demand for interdisciplinarity is a respond to the unfortunate consequences of disciplinary specialization. Or in different words, it is a matter of bridge-building.

Usually "cosmopolitan" means a world citizen, a person who feels at home across cultural borders. The use of the word above refers to a person who feels at home across borders of disciplines. Disciplines like cultures are historical constructions. In the same way that it is inexpedient for the world citizen to ignore the existence of different national cultures, it is inexpedient to ignore the existence of disciplines and differences in disciplinary cultures, if one wishes to work across disciplines.

Hence, it is important to consider the question of "*why* interdisciplinarity?" before asking "*how* interdisciplinarity?" Interdisciplinarity is, besides being a buzzword, also a fuzzy term – a fuzzword. Therefore, as always when buzz-fuzz-words are in play, one is best off with being very aware of which purpose the word serves in the given context. In this section the main message has been that interdisciplinarity is requested for different reasons and with different intentions. In the previous sections the message conveyed was that there are different kinds of disciplines. However, the circumstance that there exist different kinds of disciplines and the circumstance that there exist different kinds of interdisciplinarity are not disjunct phenomena; they are linked to each other through the degree to which a discipline is associated with a specific way of perception. This is what we address in the remaining sections of the chapter.

Competences and Way of Perception

A Danish professor of philosophy, Niels Egmont Christensen, once wrote in the foreword to a textbook on European philosophy (Schjelderup and Winsnes, 1963), "one cannot learn philosophy, at most one can learn how to *philosophize*" (ibid., 5, our translation and emphasis). That is to say, it is *not* the trivial knowledge of philosophy and the history of philosophy that is the aim. Although such knowledge is surely necessary, it is only of limited value by itself, if it is not used to communicate independent and autonomous thoughts, i.e. the outcome of the competence of philosophizing.

[21] At Roskilde University, for example, a double major may involve an integrated master's thesis in both disciplines.

Hence, philosophy is an example of a discipline which has its own way of perception, its own way of perceiving the world, its own glasses to look through (see Figure 1). In the view of Niels Egmont Christensen, philosophy is a discipline which is more strongly carried by its associated way of perception. This is not a unique situation for the discipline of philosophy. It is also the case for mathematics, as evident from the many discussions of "mathematical thinking" and "mathematizing," etc. (see e.g. Freudenthal, 1991). It is surely the case for the discipline of physics and for that of geology too, as illustrated in the example of educating oil experts. And other disciplines could be mentioned as well (but it is not our intention to make an exhaustive list). Although all disciplines have an associated way of perception, most disciplines are not primarily defined by this, but to a higher degree by the mastering of competences associated with skills and areas of knowledge. This point is best illustrated through a couple of examples.

Firstly, for both electrical engineers and physicists Maxwell's equations are central. They are the basic equations governing electromagnetic phenomena. But even though the equations may look the same on the blackboards of both the electrical engineers and the physicists, the two disciplines do not perceive the same when they look at the blackboard. For the electrical engineers, Maxwell's equations are a tool among other tools to design, for example, microphones. For the physicists, microphones are an illustration among other illustrations of the law of induction as part of Maxwell's equations. The electrical engineers focus on the microphone as a concrete whole; it must function. The physicists focus on the law of induction as one aspect of the microphone, which is of general interest as a law of nature. (For that reason, communication between electrical engineers and physicists is not always easy, when the topic is that of microphones and Maxwell's equations.)

Secondly, one may ask why the same topics, thermodynamics and quantum mechanics, normally are treated so differently in the teaching of physics and chemistry at the university level. In order to answer such a question, one must be aware of the origin of the two disciplines. Physics has its origin in the natural philosophy, which among other places can be seen from the title of Newton's *Philosophiae Naturalis Principia Mathematica* from 1687. Since then the ambition of physics has been to find the essential laws of nature in a mathematical form. Next to alchemy, chemistry has its roots in practical problems within agriculture, mining and medicine. And these diverse geneses are still reflected in the two disciplines today. Chemists are brought up with their thinking having to function in practice. Usually the essential laws of physics are far from sufficient to solve the concrete problems of the chemists. For that reason, chemistry must develop their own more application-oriented laws and recipes. The basic elements and the applicational elements merge more in chemistry than in physics. Chemists are

committed to focus on the concrete wholes, while physicists to a higher degree are in pursuit of aspects that invite to generalization. The example of the electrical engineers illustrates the difference between a discipline which is more strongly carried by a certain way of perception, i.e. physics, and one, i.e. electrical engineering, which more relies on a mixture of a way of perception, trained skills and practical knowledge. This second example illustrates that there are different mixtures of competences defining a discipline. The physicist's way of perception more or less defines his or her core competence; being a chemist to a higher degree also includes competences related to skills and knowledge areas.

So far we have tried to illustrate what we mean by ways of perception as a disciplinary competence ingredient. Surely it is not the only ingredient which defines disciplinary competences. Schematically we may identify three types of competences:

- *Competences relying on a specific way of perception*
- *Competences relying on developed and trained skills*
- *Competences relying on scholarly knowledge*

Now, this is not to say that a discipline necessarily falls into only one of the types. The three types of competences are not meant to be disjunct. Nor are they meant as a way of pigeonholing disciplines. What they *are* meant to do is to emphasize that there are different kinds of *disciplinarities* depending on different mixtures of the competences mentioned.

Linking Interdisciplinarity and Competences

To recapitulate, disciplines may in principle be borne by the three types of competences listed above. The conditions for interdisciplinary collaboration between disciplines depend on the mixture of the three types of competences which are characteristic for the involved disciplines. If we imagine a collaboration between, say, two disciplines, then the conditions, as well as the potentials for the collaboration, are very different between, for example: two disciplines mainly defined by competences mainly defined by type 1; between a discipline mainly defined by competences of type 1 and one mainly of competences of type 3; and between two disciplines both mainly defined by competences of type 3.

For collaborations between two disciplines both mainly relying on competences of type 1, i.e. heavily resting on a specific way of perception, disciplinary *integration* makes little sense. The same may be the case for collaborations between two disciplines both mainly relying on competences of type 2. As an illustration of this we may think of the skills of horseback riding and swimming, respectively. Both swimming and horseback riding are

important competences for, say, an American Indian to possess – at least as we know him from Hollywood movie productions. His need for integrating these two competences to one unified skill is not very present. The two competences complement each other as being important at the same time. But they do not allow themselves to be meaningfully integrated. The same is the case for collaborations between – or integration of – two disciplines both relying on competences of type 1. The potentials here lie in the third kind of interdisciplinarity, i.e. interdisciplinarity as understanding and overview across disciplines. The example we gave was that of exploring for oil, where both the competences and ways of perception of physicists and geologists were needed. That is to say, the kind of experts needed was "disciplinary cosmopolitans" feeling at home in both physics and geology and possessing the competences and ways of perception of these two disciplines *at the same time*. However, educating an expert, both mastering to think as a physicist and as a geologist, is not the same as in some way merging physics and geology.

For collaborations between two disciplines mainly resting on competences both of type 3, i.e. scholarly knowledge, the same kinds of barriers usually are much lower. Despite different bases of experience, such disciplines to a higher degree make use of a common (or everyday) language. For this reason, interdisciplinarity of the second kind i.e. interdisciplinarity as the integration of disciplines focusing on a certain practical area of problem and application, is possible to a larger extent. While language and perception barriers between two disciplines both mainly resting on competences of type 1 result in a need for double socialization as an arrangement of bridge building between disciplines, the barriers for an interdisciplinary collaboration between two disciplines mainly resting on competences of type 3 are typically of a more organizational and socio psychological nature. Olsen and Pedersen (1997) refer to this situation as *objective disciplinarity* (in Danish: "saglig faglighed"). That is to say, the objective is to deal with a societal problem in a serious manner. This may imply interdisciplinarity as a means in which case breaking down the disciplinary borders to have the researchers come up from their trenches and perform objective disciplinarity is needed.

For collaborations between a discipline mainly resting on competences of type 1 and a discipline mainly resting on competences of type 3, one must first and foremost be aware of the asymmetry in such collaboration. The collaboration is typically of the first kind, i.e. interdisciplinary driven by the demand for specialization. Examples are mathematical economics and bioinformatics; where in mathematical economics the focus is on applying mathematics to economic problems without leaving the economical scene, and where bioinformatics is about applying informatics to biological problems without leaving the biological scene. The disciplines mainly resting on competences of type 3 define the problems. And one may ask whether the ways of perception of the disciplines relying on competences of type 1 can

assist with substantial contributions. If the disciplines mainly relying on type 1 competences define the problems illustrated by means of examples taken from the domain of disciplines mainly relying on type 3 competences, the activities will normally fall within the former disciplines of the two.

Epilogue: Revisiting Jantsch's Taxonomy

Based on our discussion and analysis above, we are now able to relate this to the old taxonomy of Jantsch (1972), which has often been used when addressing interdisciplinarity in an educational context. Jantsch's taxonomy involves a spectrum ranging from (1) multidisciplinarity to (5) transdisciplinarity. [22] The former resembles the complete separation of disciplines that we already find in the educational system; the latter resembles interdisciplinarity to such a degree that the individual disciplines become invisible, e.g. when biology and chemistry become biochemistry, and so on. The intermediate levels of interdisciplinarity are (2) pluridisciplinarity, (3) crossdisciplinarity, and (4) interdisciplinarity (proper). Pluridisciplinarity is when disciplines cooperate about a common theme, e.g. "the 1920s" where each discipline treats this time period from only its own perspective. Crossdisciplinarity is when one or several disciplines act as *supporting* disciplines for another, e.g. when biology will rely on mathematical methods to model a purely biological problem and possibly also draw in some ethics from philosophy to clear the humanistic aspects. Interdisciplinarity (proper), in Jantsch' terminology, is when all involved disciplines are subject to a common outer principle or problem and focus moves to these

From our point of view, Jantch's taxonomy is mainly relevant for interdisciplinary collaborations between disciplines mainly resting on competences of type 3, partly relevant when disciplines mainly rest on competences of type 2 are involved, but of very little relevance when it involves disciplines mainly resting on competences of type 1, i.e. those strongly connected to a specific way of perception. The taxonomy neither takes into account the different types of competences defining the different disciplines, nor the related different reasons for interdisciplinarity. Take again the oil example. Jantsch's category of interdisciplinarity (proper) cannot provide an answer to what was needed in the oil example, the reason being that the two demanded ways of perception of physics and geology, respectively, are so different. As pointed to earlier, the need is not so much to create a specialization of exploring for oil in the borderland between geology and physics as it is to educate people simultaneously being able to manage the ways of perception of both the physicist and the geologist.

[22] Jantsch also operates with (0) disciplinarity.

The means for overcoming the shortcomings are exactly to take into account the involved disciplines' dependency on a certain way of perception.

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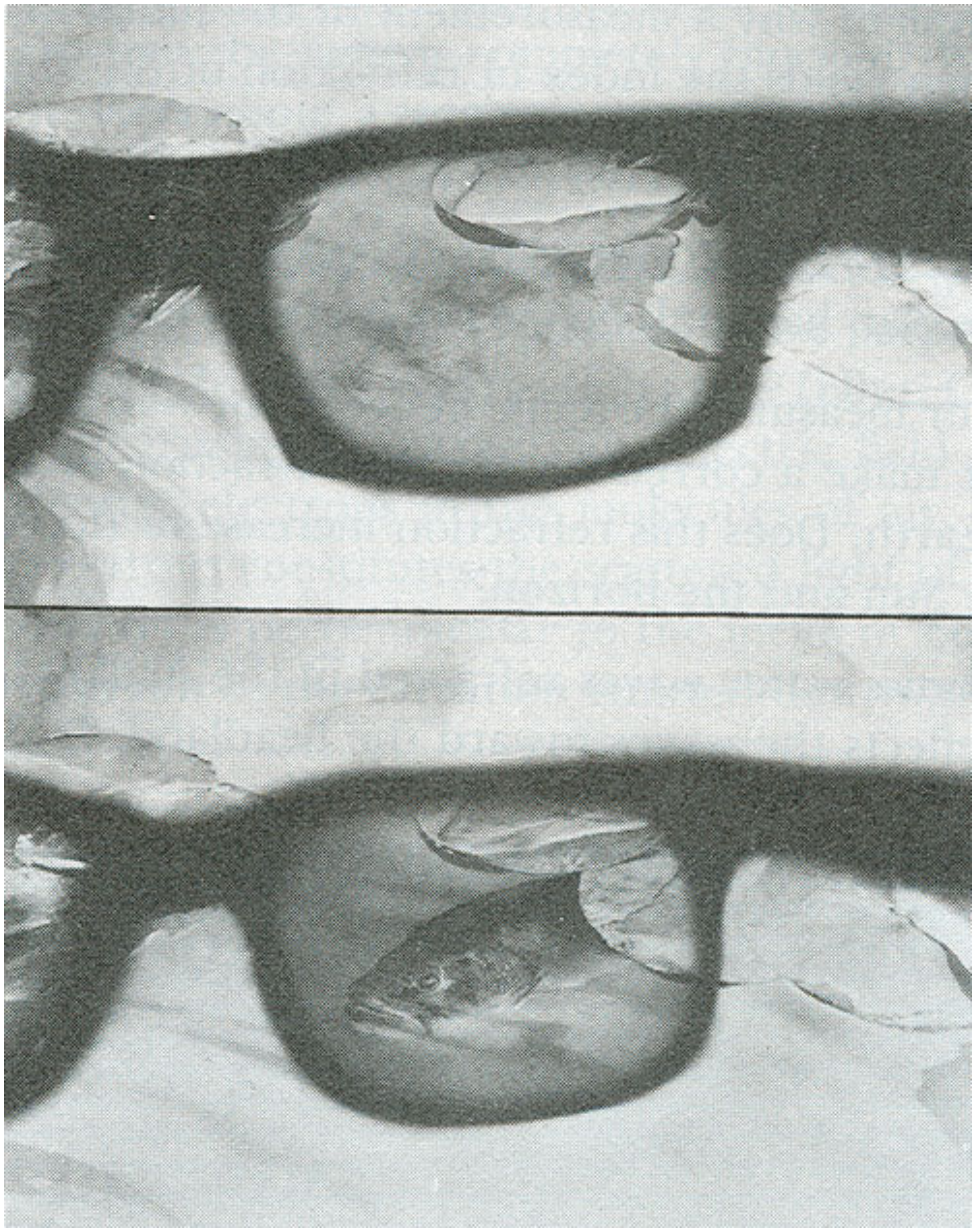

Fig. 37.32 When viewed through ordinary sunglasses (top), the fish is almost completely hidden by the glare reflected by the water. When viewed through Polaroid sunglasses (bottom), the fish becomes visible because the glare is reduced.

“Modsætningen til at overse er at se bort fra”

Artiklen her var, som den foregående, længe under vejs. De er begge dele af et oprindeligt for omfattende artikeludkast med budskaber i for mange forskellige retninger. *Nordic Studies in Mathematics Education*, hvor artiklen har været trykt, går også under forkortelsen *NOMAD*.

Disciplinary competence descriptions for external use

JENS HØJGAARD JENSEN AND UFFE THOMAS JANKVIST

The article addresses the need for competence descriptions of disciplines as a means for fostering more productive communication between different disciplines and between the disciplines and their surroundings. It is argued that the usual competence descriptions devised for use within a discipline itself, e.g. in relation to teaching and learning of the discipline – so-called competence descriptions for internal use – are not the best means to achieve this. The same is true for the general, non-disciplinary competence descriptions. Instead, specially devised disciplinary competence descriptions for external use are called for. Our main illustration is a competence description of mathematics for external use devised so that it can support the dialogue about justification of mathematics education between the discipline's practitioners and its recipients. This description for external use is counterposed with one for internal use i.e. that of the Danish KOM project. It is also counterposed with a competence description for external use for physics, taking into account the different justification problem of physics education. Together these two descriptions showcase how competence descriptions of disciplines for external use may support interdisciplinary collaboration and division of labor in the educational system.

The point of departure for this article is several years of experience with being part of the mathematics and physics "minority culture" at Roskilde University. The university is dominated by much larger institutes of the humanities and social sciences as compared to the natural sciences, of which mathematics and physics are smaller parts. Still, Roskilde University has a declared purpose – with which we strongly agree – of aiming at interdisciplinarity in both research and teaching. This circumstance has forced us, over many years, to consider and (unsystematically) investigate the collaborative roles of the disciplines of mathematics and physics in relation to other disciplines (also across faculties).

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Our accumulated experiences are that disciplines in general, and mathematics and physics in particular, are perceived by their practitioners as certain *ways of perception* (Jensen & Jankvist, 2018) built up over years of training, enabling a practitioner to perceive relationships that could not have been conceived otherwise. At the same time, we – the practitioners – often come to regard other disciplines, i.e. those which are not our own and which we therefore do not comprehend to the same extent, as merely being collections of tool-oriented recipes. For this reason, it is a challenge to find means for communicating meaningfully about disciplines between practitioners of these and recipients. A well-known example of this challenge is that of supporting disciplines at university level, e.g. supporting mathematics classes in a physics program or supporting chemistry classes in a medicine program. Oftentimes it is difficult to reach agreement on the balance between fetishisation of the supporting discipline in its own right, thus making its potentially valuable aspects almost disappear through instrumental integration. One contribution to remedy these difficulties of communication might be to exemplify and name such (only partially recognized) perspective differences.

A particularly and rather severe version of such communication difficulties may, according to our judgement, be found in connection with the teaching of mathematics (at all educational levels). Bringing matters to a head, the practitioners of mathematics often perceive the teaching of this as related to the science of pure mathematics, while the recipients (students, the educational system, and society) rather seem to expect a teaching directed at solving real life problems, i.e. something much closer to mathematical modelling. Since pure mathematics and mathematical modelling are two qualitatively different activities – e.g. consider the crucial diffe-rence between the roles of counterexamples in the two activities – this makes up a severe issue of matching of expectations. In this article, we shall attempt to articulate this issue by means of what we have named *a competence description of mathematics for external use*, and discuss the potential value of one such. It is, we argue, precisely the *competences*, as opposed to the syllabus, that are perceived differently across the gap of communication between the discipline of mathematics' practitioners and its recipients.

As the reader no doubt will know, in recent years the word "competence" (or competency) has become quite a buzzword within educational research (e.g. Han, 2010). Under the OECD auspices, work has been carried out to produce *non-disciplinary competence descriptions* for use in overall assessments of national educational systems, addressing the question of: Which competences, generally speaking, do people and society (the recipients) need the educational systems to deliver? (OECD, 2001).

At the same time, another OECD initiative, PISA, provides competence descriptions for the individual disciplines (OECD, 2013) – competence descriptions which have also been used to develop the teaching in these disciplines (subjects). By describing the subjects in terms of competences, a tighter connection is strived for to the declared goals of the subject as well as better "vertical communication" regarding the subject's place in the educational system – better than that which may usually be reached by means of syllabus descriptions. PISA's framework of mathematical competencies from 2000 to 2018 was essentially a modification of the Danish mathematical competencies framework as described in the so-called KOM-project (Niss & Højgaard, 2011; Niss & Jensen, 2002). Such descriptions may be referred to as *disciplinary competence descriptions for internal use*, since they also serve the purpose of facilitating communication between various practitioners within the discipline. Disciplinary competence descriptions for external use, on the other hand, serve the goal of providing "horizontal communication" between the practitioners of the discipline and the recipients, and thus assist in promoting collaboration.

We are aligned with the KOM-project in its efforts to augment the description of disciplinary content (syllabus) with disciplinary competence descriptions. Nevertheless, in this article we shall argue and substantiate why the KOM-project's competence description for internal use is not sufficient for the purpose of matching of expectation between the discipline of mathematics' practitioners and its recipients. Table 1 below sums up our perception of the relationship between disciplinary competence descriptions for internal and for external use, respectively, in terms of their differences in purpose and characteristics.

Table 1. *Purposes and characteristics of disciplinary competence descriptions for internal and external use, respectively*

	Purposes	Characteristics
For internal use	An *alternative* to *syllabus* descriptions.	Competence descriptions *of* the discipline.
For external use	Positioning the discipline in educational contexts. *Justification* of the discipline.	Description of the discipline's potential *contributions* to people's need for competence development.

The outset for this article is *the justification problem* of mathematics education. Still, we have chosen the title of the article to be "disciplinary competence descriptions for external use" – i.e. what might appear to be a somewhat general title. The reason for this is that in our opinion the

justification problem of mathematics, besides being distinctly alarming, constitutes an example of an issue which also exists within other subjects, to a lesser or greater extent.

Besides the competence description of mathematics for external use, we have also constructed a competence description of physics for external use. This partly serves the purpose of illustrating how competence descriptions of disciplines with a different kind of justification problem than mathematics must be devised according to that. At the same time, the comparison of the two descriptions invites to a dialogue between the two disciplines regarding their internal distribution of roles in relation to teaching mathematical modelling as an illustration of how competence descriptions for external use may be utilized within interdisciplinary collaboration in the educational system. We also consider this to be an example of how disciplinary competence descriptions in interdisciplinary contexts may contribute to communication across disciplines at large. It should be mentioned that by counterposing our competence description of mathematics for external use with a corresponding one for physics, the context is enhanced in direction of *utilization*, while the analogous description of physics at the same time provides some backlight. Such backlight could equally well have been provided by counterposing mathematics with, say, the discipline of history, which then might have brought out more cultural aspects of the discipline of mathematics. In this article, however, we have deliberately chosen a more utilitarian focus, since this particular aspect of mathematics is the one often provided when it comes to justifying mathematics education and its relevance in society. In order to situate our pending disciplinary competence descriptions for external use, we now briefly turn our attention to the so-called justification problem of mathematics education.

The justification problem of mathematics education

Blomhøj (2001) points to the fact that the problem of justification of mathematics education has both an objective and a subjective side. The objective side concerns the reasons for mathematics education placed in a societal context. The subjective side, on the other hand, concerns the single individual's (e.g. teacher or student) sense making of participating in mathematics education. Of course, such a distinction between the objective and subjective may be made concerning the justification of any subject. Nevertheless, the situation for mathematics is a special one in terms of objectivity and subjectivity; a situation sometimes referred to as the *relevance paradox* (Niss, 1994). The objective side of the paradox has to do with "the unreasonable effectiveness of mathematics" both in the

natural sciences (Wigner, 1960) and in engineering and computer science (Hamming, 1980). That is to say, mathematics applies to – and is applied in – a wide range of extra-mathematical subjects and practice areas, and in particular physics. This use of mathematics permeates society at large, in its past and present functioning and evolution as well as in its future development. The use of mathematics in technology constitutes one example of this, both the material technology (physical objects and systems, e.g. computers and other microelectronic devices) and the immaterial technology (computer software, codes, geographical coordinates, calendars, money transactions, graphical representations, measurements of time, space, weight, currency, etc.) (Niss, 1994). Other examples, also of an immaterial nature, are the various decision-making and controlling processes which take place as part of the infrastructure of a society and various forms of descriptions and predictions, e.g. about the weather and climate. The application of mathematics to these extra-mathematical areas is brought about by mathematical modelling, including the building, usage, and validation of such models. However, the embeddedness of mathematics into mathematical models and (other) immaterial technologies, as well as the further potential embedding of these into material technologies, brings about the subjective side of the paradox, namely that the mathematics in society becomes invisible – hence irrelevant – to us. Or as Philip J. Davis, co-author of *The mathematical experience* (Davis & Hersh, 1981) sees it: "[...] it's invisible to people because it's in programs, it's in chips, it's in laws [...]. So you don't see it, and if you don't see it, you don't think it's there" (Interview with Davis in Jankvist & Toldbod, 2005, p. 17). This is exactly what constitutes the relevance paradox: "the simultaneous objective relevance and subjective irrelevance of mathematics" (Niss, 1994, p. 371).

A common belief – also among the teachers of mathematics – is that society buys the ability to solve problems using mathematics by teaching mathematics in its own right, i.e. pure mathematics. This enforces the relevance paradox by regarding a necessary condition to also be sufficient, i.e. the insufficiency in meeting the relevance paradox is not recognized. At the same time the instrumental perception of mathematics as a discipline paves the road for disciplinary relational understanding to be outsourced to various digital technologies, not least computer algebra systems (CAS) (e.g. Jankvist & Misfeldt, 2015). Collectively speaking, both the mathematics teachers as well as their surroundings need an understanding of the fact that bringing mathematics in play in extra-mathematical problem solving situations demands yet a set of competences besides being able to master the inner aspects of the discipline itself (see also Jensen, Niss & Jankvist, 2017). And it is needed

that the surroundings understand that mathematical problem solving and modelling, as something different than actions of routine, in the extra-mathematical domain necessarily presupposes a non-algorithmic and creative relationship to mathematics in its own right. Without such understandings, among the discipline's practitioners as well as its recipients, it does not appear sure that mathematics will continue to be the major subject in school that it is today.

A competence description of mathematics for internal use

In the Danish KOM-project, Niss and Højgaard (2011) generally define a person possessing a competence within a discipline, as taken to be someone who is able to master essential aspects in relation to that discipline effectively, incisively, and with overview and certainty of judgement. They talk about the discipline of mathematics as being associated with two such overall *competences*: (i) the ability to ask and answer questions in and with mathematics, and (ii) the ability to deal with mathematical language and tools. Each of these competences is made up of a set of mathematical *competencies* (notice the distinction used between competen*ce* and competen*cy*). While "mathematical competence comprises having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role", "a mathematical competency is a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, p. 49). The first overall competence (i) consists of the four competencies of mathematical thinking, problem tackling, modelling, and reasoning. The second (ii) consists of the four competencies of representing, symbol and formalism, communication, and finally that of aids and tools. Each of the eight competencies has both an analytic side involving understanding and examining mathematics, and a productive side involving carrying it out. The modelling competency, for example, consists of the ability to analyze the foundations and properties of existing models and to assess their range and validity, on the one hand. On the other hand, it involves being able to perform and utilize active modelling, including mathematizing, in given extra-mathematical contexts and situations.

In addition to the eight mathematical competencies, the KOM-project also describes three types of "overview and judgement" (OJ), which are to be thought of as "'active insights' into the nature and role of mathematics in the world" (Niss & Højgaard, 2011, p. 49). Niss and Højgaard state that "these insights enable the person mastering them to have a set of views allowing him or her *overview and judgement of the relations between*

mathematics and [...] *nature, society and culture*" (p. 73). These are: (OJ1) the actual application of mathematics in other subject and practice areas; (OJ2) the historical development of mathematics, both internally and from a social point of view; and (OJ3) the nature of mathematics as a subject.

The KOM-project's competence description for internal use, with its eight competencies and three types of overview and judgement, makes up a clear invitation to non-mathematicians who are genuinely interested in understanding what mathematical thinking is composed of. But with mathematical modelling as only one of eight competencies, and not as *the* central component corresponding to the expectations of society, the description is not optimal as a basis for dialogue between the discipline's practitioners and its recipients.

A competence description of mathematics for external use

The purpose of competence descriptions for external use is to function as a platform, not for communication between practitioners internally, but for communication between practitioners of the discipline and its surroundings in the form of students, practitioners of other disciplines, authorities, and society in the widest sense. In our formulation of the competence description of mathematics (and later that of physics) for external use, we have confined ourselves to looking at three overall competences; competences which we find may span the potential space of dialogue. Taking into account how easily words may be adjusted to the reader's universe, and hence also unintentionally misinterpreted, we shall provide illustrative examples for each of the competences to brace the formulations.

For mathematics, we have formulated a description directed at the schism between, on the one hand, the kind of focus which mathematicians naturally have on the distinctive characteristics of the discipline, manifested by "pure mathematics", and on the other hand, the surroundings' expectations that teaching actually trains students in tackling extra-mathematical problems by means of mathematics. As is the case with the KOM description of mathematical competencies for internal use (Niss & Højgaard, 2011), competence descriptions of mathematics for external use may not be so sharply defined and described that overlaps will not occur. The competences of the descriptions for internal use and external use, respectively, are bound to be connected, also in terms of overlap, and certainly the competences in the descriptions for external use cannot be developed in isolation from those in the descriptions for internal use. For the description for external use, we have identified the following three mathematical competences:

1. Being able to think as a mathematician
2. Being able to relate and respond to actual applications of mathematics
3. Being able to bring mathematics in play in extra-mathematical settings

The first competence is one that encompasses seven of KOM's mathematical competencies: mathematical thinking, problem tackling, reasoning, representing, symbol and formalism, communication, aids and tools as well as aspects of OJ3 on the nature of the subject of mathematics. (In relation to teaching, it would hopefully also include aspects of OJ2.) The second and third of the mathematical competences in the description for external use are related to the modelling competency in the KOM description. Recall that the modelling competency comprises two parts; one analyzing part and one performing part. Under the assumption that practically all uses of mathematics for extra-mathematical purposes boil down to a use of modelling in some sense, the second competence – "being able to relate and respond to actual uses of mathematics" – then embraces the analyzing part of the modelling competency, whereas the third – "being able to bring mathematics in play in extra-mathematical settings" – corresponds to the performing part of the modelling competency. In addition to being made up of the analyzing part of the modelling competency, the second competence in the description for external use includes also OJ1 on the actual application of mathematics in other subject and practice areas. This being said, the second and third competences above can of course not be practiced without substantial possession of KOM's other seven competencies, i.e. the first competence in our description for external use. The reverse, however, may indeed be the case; students trained in problem tackling within the internal mathematical domain may not be able to model extra-mathematical situations (e.g. Ikeda & Stephens, 1998). We now illustrate the three competences of the competence description for external use with three examples.

Being able to think as a mathematician

As an example, we take the following discrete mathematics problem, which students often find intriguing: *Show that at a party, where there are at least two people, there are (at least) two people who know the same number of other people there.* To solve this problem, we consider an arbitrary person, say p, who is at the party. We define the function $K(p)$, to be the number of other people at the party which person p knows. Now,

the possible values for $K(p)$ are 0, 1, ... , $n-1$, where $n \geq 2$ is the number of people at the party. We then observe that it is impossible for both 0 and $n-1$ to be in the range of K, since if one person knows everyone else, then nobody can know no one at the party (we make an assumption here that "knowing" is a symmetrical relation). Therefore, the range of K has at most $n-1$ elements, whereas the domain of K has n elements, so K is not a one-to-one function, which corresponds precisely to what we wanted to show!

As can be seen, if analyzing our reasoning step by step, the majority of KOM's mathematical competencies come into play when providing the above chain of arguments. We boil the situation down so that we become able to apply a well-known principle from mathematics, the *Dirichlet drawer principle* (or the pigeonhole principle), which says that if $k+1$ objects or more are to be placed into k boxes, then there is at least one box containing two or more of the objects. Of course, non-mathematicians would askwhy we bother to name something so obvious as "a principle". But this illustrates exactly one aspect of being able to think as a mathematician; to take a more complex situation, e.g. that of the party, and reduce it to something to which we already know the answer, in this case the Dirichlet drawer principle.

Being able to relate and respond to actual applications of mathematics

To illustrate the second competence we take as our example models for forecasting populations. Many a country maintains a model for forecasting its population development. In such models some basic population dynamical mechanisms are rock solid, while others are subject to assumptions, estimations, etc. The Danish model for forecasting populations, in its present form, has been maintained by *Statistics Denmark* since 1977 (for a discussion, see Niss, 2000), and operates with progression through age classes. The model comprises the following rock solid mechanisms: individuals already born progress in the age classes by becoming one year older per year, provided of course that they survive; individuals may enter or leave an age class during a given year by either immigrating/emigrating or by dying; and new individuals are born by women in the fertile age classes. However, when doing actual forecasting, estimates must be made of death frequencies, net immigration figures, and not least the fertility quotients to enter the model. These are highly dependent on biological, cultural, and socioeconomic factors, making them vary from "almost solid rock" over "soft soil" to mere "mud" (using the terms of Niss, 2000). For example, over time the age classes in which Danish women typically give birth to children have shifted, e.g. due to better family planning or

because women wanted to finish their education before having children, which resulted in an unexpected drop in the fertility quotients of young women in the 1980s; something the model failed to predict. Being able to tell the difference between the fixed mechanisms in a given model and mechanisms which are based on loose estimates and assumptions (even if including historical data, as the Danish model of course does), is part of being able to relate and respond to actual applications of mathematics. Besides mastering mathematics, the competence of "being able to relate and respond to actual applications of mathematics" presupposes epistemological judgements based on experience and interactions with the domain mathematics is applied to.

Being able to bring mathematics in play in extra-mathematical settings

As an illustration of the third competence we offer the example of answering the question: *How far away is the horizon?* In order to tackle this question, we first need to conceptualize what is meant by "horizon" – from a modelling perspective, this step corresponds to what Niss (2010) refers to as pre-mathematization, i.e. crucial steps preceding the actual process of mathematization. This is illustrated by means of figure 1.

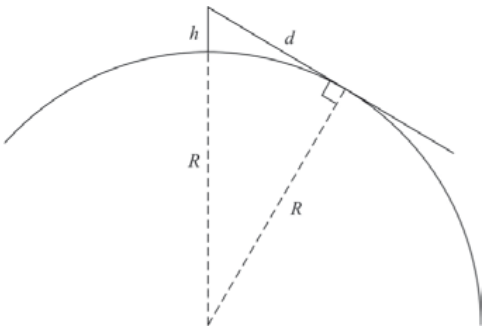

Figure 1. *An illustration of the distance, d, to the horizon, taken from the height of observation, h, to the point where the tangent line touches Earth's surface. R is Earth's radius*

We assume the height of observation, h, as being quite small in comparison to Earth's radius, R. For that reason, we shall calculate the distance, d, from the top point of h, from where we observe, to the point of the horizon as being approximately the same as the distance from the ground point of h along the surface of Earth to the point of the horizon. It has no practical importance if we by the distance to the horizon take one or another. We now see that d is given by the Pythagorean theorem

$(R+h)^2 = d^2 + R^2$, where $d = \sqrt{\left((R+h)^2 - R^2\right)} \approx \sqrt{2Rh}$

since h^2 is much smaller than $2Rh$. Now, if $h \approx 2$ meters and $R \approx 6000$ kilometers in the formula above, we get $d \approx 5$ kilometers. If, say, we put our observation post in a tower, so that h is increased by a factor 16 to 32 meters, we can immediately deduce from the formula that d is increased by a factor 4, so that the distance to the horizon now is 20 kilometers.

While the competence of "being able to relate and respond to actual applications of mathematics" requires a knowledge of the disciplinary domain to be modelled, this is not the most important component of the competence of "being able to bring mathematics in play in extra-mathematical settings". The essential thing here is not to model using a lot of problem-specific knowledge, e.g. except knowing that the Earth is spherical as in the example, but to bring mathematics to use by making continuous simplifications to the extra-mathematical problem. In the words ascribed to Albert Einstein, the idea is to make things as simple as possible, but not simpler. This is indeed unlike the competence of "being able to think as a mathematician", where the idea is to make things logically exact.

Discussion – part 1

What happens when shifting the focus from a description for internal use to one for external use is that some of the competencies in the description for internal use are grouped together as competences in the description for external use, while others are split up and differentiated. As the reader will have noticed, the proposed competence description of mathematics for external use may be seen as a merger of seven of the KOM-project's competencies into one competence (i.e. "being able to think as a mathematician"), while taking the last competency on modelling and splitting it up according to its analyzing and its productive sides (i.e. "being able to relate and respond to actual applications of mathematics" and "being able to bring mathematics in play in extra-mathematical settings"). As illustrated by our first example above, being able to think as a mathematician does involve to represent things mathematically, to apply symbols and formalism, conduct mathematical problem tackling and reasoning, communicate the solution and reasoning involved, etc. – and from an internal mathematical point of view it does make sense to distinguish these different competencies involved. But for someone outside the discipline of mathematics, doing so may cause more confusion than clarity, since all of it is being able to do mathematics and some possession of all

of these competencies is required to tackle extra-mathematical problems of society. In terms of the mathematical modelling competency, however, being able to analyze already made models is quite different from being able to build mathematical models oneself, which is also illustrated by our two examples above. Being able to relate and respond to actual applications of mathematics requires sound knowledge of the, for example, societal contexts in which these are a part. Being able to bring mathematics in play in extra-mathematical settings, however, requires practice with doing modelling and it requires "implemented anticipation" (Niss, 2010; Stillman & Brown, 2014), which refers to the modeler being able to anticipate forthcoming steps in the modelling process and "to implement this anticipation in terms of decisions and actions that frame the next step to be made" (Niss, 2010, p. 56). Both of these competences are of societal importance. Still, the recipients to a larger degree seem to expect possession of the latter than the former of the two.

The competence descriptions of mathematics for internal and external use, respectively, are not in conflict with one another. They merely stress different aspects of the discipline. The competence description for internal use guides the discussion concerning what characterizes mathematical activity, whereas the competence description for external use opens up for discussion concerning the arguments for including mathematics in a given interdisciplinary setting (in both research and teaching). It all depends on the purpose of the description (and discussion); in general, competence descriptions are directed normatively against problems at hand. The internally oriented one for mathematics is directed against *syllabusitis* (Lewis, 1972) as a problem, i.e. it constitutes an alternative to syllabi descriptions of the discipline. The external oriented one for mathematics has a purpose of bridging a potential conflict between an internal perception of the discipline and the expectations of the recipients of the surrounding world (cf. table 1 in the introduction).

Returning to the problem of justification of mathematics education, Niss and Højgaard offer two reasons why mathematics is such a dominating educational subject. Firstly, because mathematical competencies "often are necessary to be able to utilise technology in the traditional sense, and secondly, because mathematics by means of mathematical models, in itself constitutes such technology." They continue: "This completes the reasoning which from a recipient's point of view justifies the importance of mathematics teaching: An educational policy placing mathematics teaching at the front is, amplified by the labour market policy, essential for the maintenance and development of the welfare state" (Niss & Højgaard, 2011, p. 169).

A competence description of physics for external use

As the discipline of mathematics, mathematically oriented physics is subject to the previously mentioned relevance paradox. However, as can be seen from figure 2, the consequences of the relevance paradox are different for the two subjects within the dominating part of Danish upper secondary school (the classical stream, referred to as stx). The upper secondary school subject of mathematics is, as previously mentioned, threatened by an instrumental perception of the subject resulting in a tendency to outsource it to various digital technologies. But quantitatively speaking, the subject of mathematics is not threatened so far. The subject of physics, on the other hand, is. As illustrated by figure 2, physics has over a long period of time been on retreat in the Danish upper secondary school.[1] In comparison to the subject of chemistry, the retreat of physics is most likely due to its larger content of applied mathematics. Unfortunately, it is precisely this aspect of physics, i.e. the involved application of mathematics, which is perhaps of most objective relevance generally speaking.

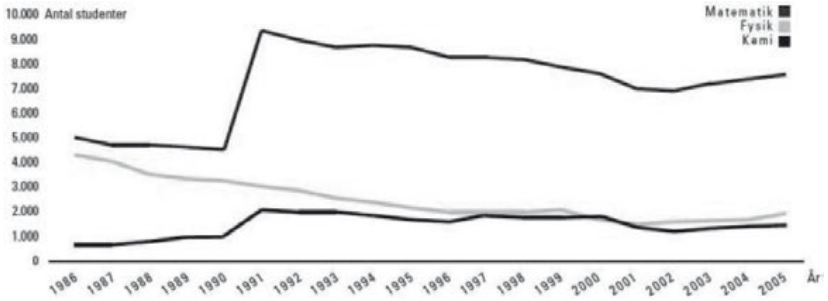

Figure 2. *Number of stx students with A-level in mathematics, physics or chemistry from 1986 to 2005 (UVM, 2006, p. 35). A-level corresponds to having the subject for three consecutive years at upper secondary school*

In order to focus on the justification problem of physics education in such situations, we thus find it relevant to devise a disciplinary competence description of physics[2] for external use as consisting of the following three competences.

1. Being able to think as a physicist
2. Being able to tackle problems in an experimental manner
3. Being able to tackle problems in a formalizing manner

These three competencies correspond to three quite diverse reasons for teaching physics. A trustworthy justification for the subject of physics as part of the educational system presupposes such nuances, i.e. that it is a subject about the nature, an experimental subject, and a mathematical subject – and the justifications for subjects about nature, for experimental subjects, and for mathematical subjects are not the same. In situations such as that of the Danish upper secondary school there is a need for an external competence description of physics, which can be used to foster communication between the practitioners of physics and the outside world regarding what is lost, when the subject of physics is phased out. The present competence description therefore heralds a review of the relative importance of the different kinds of justifications for teaching physics. Is it the character of physics as a subject about nature which makes it hard to do without? Is it physics as an experimental subject which makes it valuable? Or is it physics as a mathematical subject that constitutes the most important justification for the subject within the educational system?

Being able to think as a physicist

The acquisition of physical concepts and knowledge as a potential profit from physics education is of course important. As an example of this, think about the importance of understanding the concepts of heat and temperature. Some years ago in Denmark, measures taken to save energy resulted in a law stating that the common heating bill in apartment buildings should be distributed between apartments by means of radiator gauges. But radiator gauges do not measure heat expenditure. This requires expensive flow measurements of the radiator water. Instead they measure the surface temperature of the radiator, which is much cheaper to measure. Now, if a radiator is in some way covered up or built in, measures of the heat expenditure will quite surpass the actual heat consumption. On the other hand, money can be saved on the expense of the neighbors by ensuring a high degree of air circulation around the radiator. For that reason, radiator gauges only distribute fairly the expenses if all radiators are placed and built in similarly in all apartments – which is not the case in old apartment buildings. However, for the majority of people in Denmark the concepts of heat and temperature are more or less synonymous. Therefore, the installation of radiator gauges has overall led to energy savings. This would not have been possible had people known that they actually paid for something different than what they thought.

Thus, the competence of "being able to think as a physicist" – here illustrated by the usefulness of knowing and understanding concepts

from physics – is of course not an unimportant part of the potential profit of having been taught physics. Nevertheless, this competence, among the three making up our competence description of physics for external use, is the one which is the easiest to communicate externally.

Being able to tackle problems in an experimental manner

Communicating the second competence to people without any experimental experiences is more challenging. For example, how do you explain the crucial differences of experimental evidence for earth rays and radioactive rays, respectively? Both are impalpable and invisible. In order to understand the differences some already established sense of "scientific method" is needed. To a certain extent the usefulness of the competence is first recognized when you possess the competence. To experimental competence belongs, besides a methodical approach, also skills of measuring techniques, handling of numbers, calculating with units and assessing uncertainty. But whether focus is on the skills or on the overall experimental grip, one has to ask if these matters may not be equally or perhaps even better motivated in chemistry and biology teaching. In England the introduction of "combined science", which integrates physics, chemistry and biology, has promoted biology at the expense of physics in upper secondary school (Smithers & Robinson, 2006). In relation to developing the competence of tackling problems in a formalizing manner, this poses a problem. However, in relation to developing the competence of tackling problems in an experimental manner, this need not necessarily be the case.

Being able to tackle problems in a formalizing manner

Besides being a subject about nature and an experimental subject, physics is also a mathematical subject, which from upper secondary school level onwards trains students in being able to tackle problems in a formalizing manner (see also Jensen, Niss & Jankvist, 2017). To illustrate what we mean by this, we offer the following example of answering the question: *How does the thickness of the ice on a lake of water of 0° C grow with time if the air is freezing cold?* In order to tackle this problem, we first represent it by the simple figure 3.

Next, we analyze the physics in the problem. When the water beneath the ice is freezing, energy is released as heat. This heat is transported away through the ice to the air, allowing the freezing to continue. Our third step is to formalize this in a mathematical manner. From an area A of the ice, an amount of heat $LAdx$ is produced, when the ice is increasing its

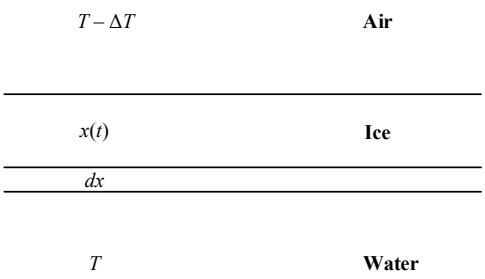

Figure 3. *Growing of ice due to freezing of water from beneath. x(t) is the thickness of the ice, dx an infinitesimal increase of x, T the temperature of the water and ΔT the decrease of temperature from water to air*

thickness with *dx*, and *L* is the latent heat of water/ice. If the increase *dx* of *x* happens during the time *dt*, the amount of heat conducted through the area *A* of the ice into the air at the same time is $(\kappa A \Delta T/x)dt$, where κ is the heat conduction coefficient of the ice. Setting equal the amount of heat produced with the amount of heat conducted away in the time period, *dt*, we get:

$$LAdx = \frac{\kappa A \Delta T}{x} dt$$

Having mathematized the problem, assuming ΔT constant, it can be solved integrating the above equation. If we measure *t* from the onset of freezing, we get:

$$x(t) = \left(\frac{2\kappa \Delta T t}{L} \right)^{\frac{1}{2}}$$

Thus, the thickness of the ice on a lake grows as $t^{1/2}$ from the beginning of the time of freezing. The heat conduction coefficient κ is defined as the proportionality constant between heat flow density and the temperature gradient. In a stationary situation the heat flow in and out of any layer of the ice must be the same. Thus, the temperature gradient has the constant value $\Delta T/x$ throughout the ice.

Discussion part 2

With the above example, we try to insinuate some of the essence in solving problems by means of formalizing. As was the case for the previous problem of the distance to the horizon, the solution to the above problem also makes use of relatively simple mathematics. Still, the problems would be difficult for the majority of upper secondary school students, because the tackling of them presupposes formalizing and

mathematization or pre-matematization (notice also that in the horizon problem, a typical perception of the physicist – or the modeler – comes into play, when disregarding the term h^2, such reasoning would not be usual for the pure mathematician). As explained, the third bullet in the competence description for external use of mathematics – being able to bring mathematics in play in extra-mathematical settings – concerns exactly this aspect. Hence, it more or less coincides with the third bullet in the competence description for external use of physics – being able to perform formalizing problem tackling. The outlined competence descriptions for external use therefore herald a dialogue between both mathematics and physics themselves as well as in relation to their surroundings. Does the teaching of mathematics deliver the kind of competence wanted by society? Is the main potential of physics to be a practice ground for bringing mathematics in play in extra-mathematical situations? Is it the teaching of mathematics or the teaching of physics which holds the largest potential to educate students to be able to tackle, for example, problems of infinitesimal calculus as that of the growth of ice on a lake's surface or problems of geometry as that of the distance to the horizon? Competence descriptions for external use can make up the departure point for such interdisciplinary dialogue.

Without dialogue between the mathematics educators and physics educators, both internally and with the surrounding world, there is a risk that the educational system's support to develop students' competences for formalizing problem solving withers away. Who picks up the gauntlet, if mathematics is oriented towards proofs and proving, and physics is oriented towards the laws of nature? Does not the responsibility for teaching this competence then fall between two stools? Who in the surrounding world is to detect such lack of responsibility before it is too late and without being made aware of it? This is to say that it is not sufficient that only a minority of the population become acquainted with physics as a mathematical subject, and that we have to consider the danger of the subject of mathematics being reduced to memorizing recipes and pushing buttons. Despite the different justification problems of mathematics education and physics education, both the quantitative success and qualitative letdown of mathematics, and the subject of physics' retention of qualitative standards with the quantitative letdown as a consequence, is connected to a too weak understanding of mathematical modelling. It is realized as important for society, but not realized as a distinctive competence.

As initially mentioned, many of for example OECD's competence descriptions do not relate to a discipline (subject) at all, but to cross-curricular student activities, teaching methods, school organization, etc. And surely subjects have many common challenges of competence

development in common, e.g. dedication, precision, critical sense, etc. Such general competences are not the topic of interest in this article. Rather our focus is on the communication between disciplinary practitioners and their surroundings in relation to the various disciplines' specific potential contributions to students' competence development. From experiences at Roskilde University we know that it is difficult to carry out common discussions about interdisciplinary collaborative teaching (Jensen & Jankvist, 2018). It is simply difficult to have discussions across disciplines between their disciplinary practitioners, because one views his or her own discipline from the inside and other's disciplines from the outside.

The term "interdisciplinarity" is often used within quite different contexts given the term's different meanings (Jensen, 1991, 2012). It is used in the process of establishing new disciplines on the border between already established disciplines. It is used in the endeavors to orient already established disciplines towards various functions and problems in different applicational contexts. Finally, it is used when there are calls to cognitively bridge the divide between disciplines. We imagine that competence descriptions for external use may be of assistance in all such cases, although in particular in situations calling for some kind of "bridge building".

In a sense, the preparation of competence descriptions for external use may be compared to "speaking to blind people about colors". Disciplines may to varying degrees be constituted by specific domains of knowledge, specific methods, or by specific ways of perception (Jensen & Jankvist, 2018). The more a discipline's center of gravity is made up by a specific way of perception, the more difficult external communication becomes; the character of the specific way of perception is difficult to comprehend, if it is not itself mastered to a certain extent. Both mathematics and physics are disciplines which to a very large degree are constituted by their different, fundamental ways of perception. Hence, mathematics and physics are among the disciplines that are particularly challenged when involved in interdisciplinary activities. One of the most challenging aspects is to achieve insightful communication with other disciplines concerning their contributions in planning educational programs. Our competence descriptions of mathematics and physics for external use are thus also meant as a door opener in this situation. Is it, for example, physics, chemistry or biology which first and foremost are to deliver the competence of "being able to tackle problems in an experi-mental manner"? Or is it in the mathematics teaching or the physics teaching that the competence of "being able to bring mathematics in play in extra-mathematical settings"/"being able to tackle problems in a formali-zing manner" is to

be developed? Competence descriptions of disciplines for external use may be means for dealing with such questions.

Concluding remarks

In this article, we have augmented the general, non-disciplinary competence descriptions of what purpose the educational programs might possibly serve and the disciplinary competence descriptions for internal use with two examples of different competence descriptions – descriptions for use in the discussion of disciplines' potential contribution to its surroundings, not least in interdisciplinary collaborations for the purpose of fulfillment of the educational programs' overall purpose. We have named these "disciplinary competence descriptions for external use". While the purpose of the disciplinary competence descriptions for internal use is to ease the "vertical communication" within the disciplines, the purpose of disciplinary competency descriptions for external use is to ease the "horizontal communication" between disciplines and their surroundings, i.e. between practitioners of the disciplines and the recipients.

Let us illustrate with a metaphor: As a precondition for well-functioning international cooperation, it is important that the national cultures are open to mutual understanding. And this demands much more than formal tolerance. It is necessary to consider how one declares oneself to the others. This is also the case with interdisciplinary collaboration – in all its different shapes. Competence descriptions for external use may contribute to qualifying considerations regarding how the nature of interdisciplinary collaborations depends on the types of involved disciplines.

There is, we argue, a need for different degrees of detail and accentuation of involved sub-competences depending on whether the target group is made up of the practitioners of the discipline or of collaborators from other disciplinary areas. The most defining needs for externally oriented competence descriptions arise once disciplines are to contribute to complex abstract wholes outside the disciplines themselves. For example, due to a lack of mutual meta-understanding it is difficult to find a balance between elimination and disciplinary fetishism once one discipline is to act as support for another discipline. Or generally speaking, when entire educational programs are planned involving several disciplines. Returning to the metaphor, as in international collaboration this requires a common and mutual understanding of each other's cultures. Disciplinary competence descriptions for external use may assist in providing just that.

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Notes

1 Similar international tendencies are reported by Jørgensen (1998) and Smithers and Robinson (2006).

2 Of course, a competence description of physics for internal use could also be given, but that is not our purpose here.

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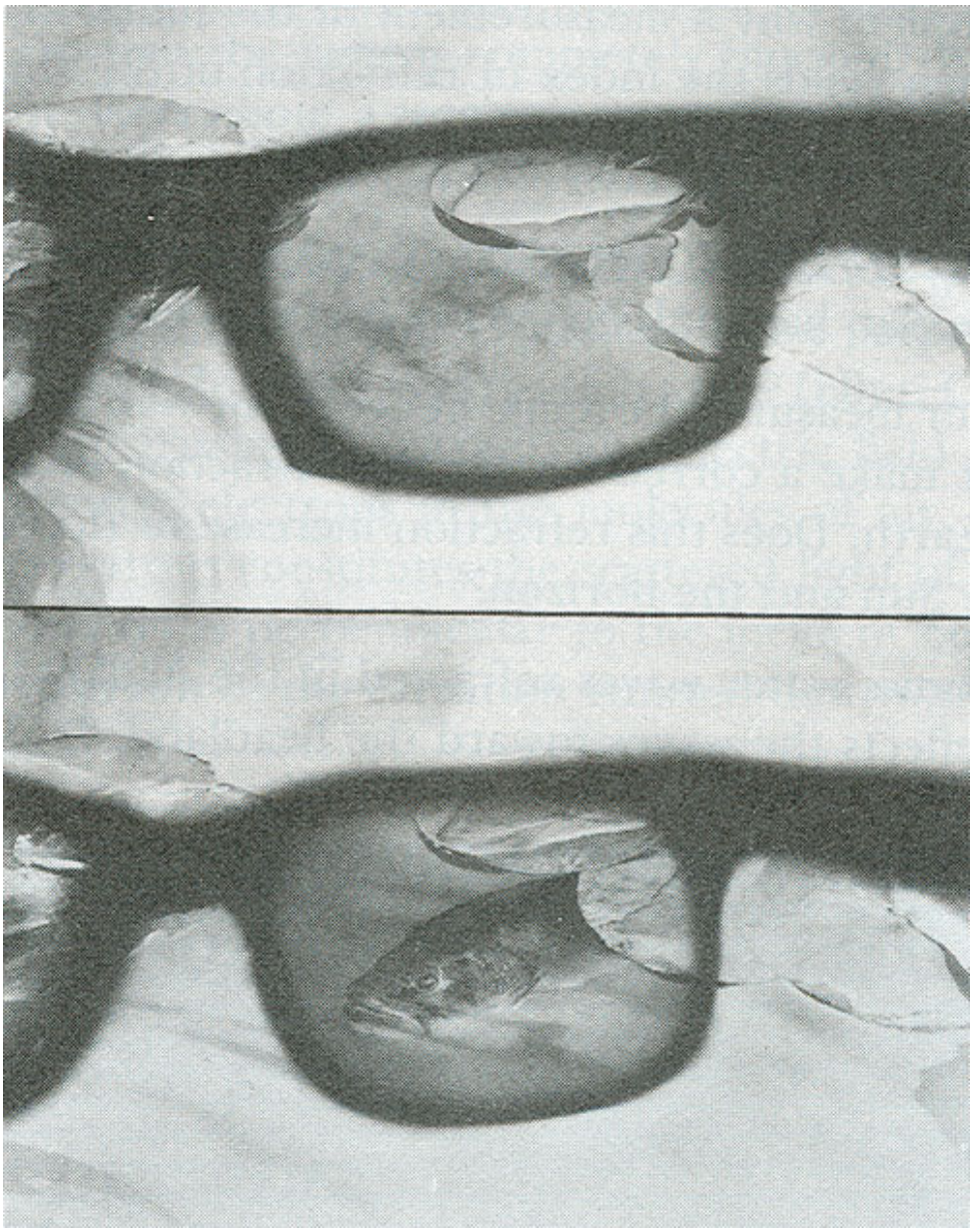

Fig. 37.32 When viewed through ordinary sunglasses (top), the fish is almost completely hidden by the glare reflected by the water. When viewed through Polaroid sunglasses (bottom), the fish becomes visible because the glare is reduced.

“Det sete afhænger af øjnene, der ser”

Tina og jeg sled i det ved udarbejdelsen af den følgende artikel om dimensionsanalyse. Vi blev bedt om i det væsentlige at undgå formler i artiklen. Men hvordan skrive en artikel om udledning af formler uden illustrerende brug af formler? Resultatet blev en introducerende tekst uden mange formler med et elektronisk appendiks med fem eksempler, der illustrerer, hvordan man faktisk gør, når man udleder formler ved hjælp af dimensionsanalyse.

FYSIKERNES HEMMELIGE VÅBEN

Fysikernes form for bogstavregning gør det muligt at udlede nye formler ved dimensionsanalyse. Dette hemmelige våben kan andre fagområder også have glæde af, da det kan give en dybere forståelse af sammenhænge, der ellers blot er erfaringsbaserede.

Om forfatterne

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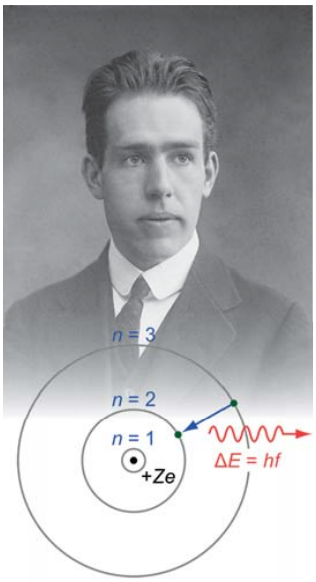
Det er indlysende, at ligningen 18 får = 18 mursten er noget vrøvl, selvom der står 18 på begge sider af lighedstegnet. På samme måde er 18 sek = 18 m noget vrøvl. I gymnasiet skal man derfor i fysiktimerne lave enhedskontrol af sine udregninger. Hvis en udregning ikke ender med samme *slags* enheder på begge sider af lighedstegnet, er der begået en fejl, som man bør finde og rette. Den ene side af lighedstegnet må for eksempel ikke have enheden m/sek, medens den anden side af ligningen har enheden m/sek^2. Det afgørende ved enhedskontrol er, at enhederne på hver sin side af lighedstegnet repræsenterer samme slags fysiske størrelse, for eksempel den fysiske størrelse hastighed. Eller sagt på en anden måde: De to sider af ligningen skal have samme dimension, for eksempel dimensionen hastighed (som er en længde divideret med en tid). Når fysikere laver dimensionsanalyse udnytter de konsekvenser af dette krav – ikke blot til enhedskontrol, men også til teoretisk at udlede nye formler ud fra kendte formler og nye antagelser.

Da Niels Bohr i 1913 udviklede den model for brintatomet, som skulle



blive startskuddet til kvantemekanikken og forståelsen af elektronernes opførsel i atomerne, lod han sig således lede af dimensionsanalyse. Og der er mange spørgsmål indenfor fysikken, man kan tænke sig til svaret på ved hjælp af dimensionsanalyse – uden at udføre et eneste eksperiment. For eksempel, hvor lang tid det tager et timeglas på Månen at tømmes.

Det er imidlertid ikke kun fysikere, der har glæde af dimensionsanalyse. Også specialiserede ingeniører, for eksempel dem, der laver hydrodynamiske og aerodynamiske modelforsøg, bruger dimensionsanalyse som et vigtigt værktøj. Men slagkraften af dimensionsanalyse er skjult for mange, der kunne have udbytte af tankegangen - også uden for fysik og specialiserede ingeniørfag. Artiklen her er ment som en introduktion til dimensionsanalyse og tænkningen bag.

Symboler i fysik og matematik

I matematik og fysik udtrykker vi os ved hjælp af symboler. Talsymboler, eksempelvis 18, når vi regner med tal (aritmetik). Bogstaver som pladsholdere for tal, når vi regner med bogstaver i matematik (algebra). I fysik regner vi, som i matematik, også med bogstaver. Her er bogstaverne imidlertid pladsholdere for fysiske størrelser, ikke for tal. Kort fortalt er dimensionsanalyse relevant, når vi regner med formler, hvor bogstaverne i formlerne repræsenterer fysiske størrelser, ikke tal som i matematikken.

For at tydeliggøre tænkningen bag dimensionsanalyse vil vi groft skitsere de tre abstraktionsspring ved brugen af symboler i matematik og fysik gennem menneskehedens historie:

Pythagoras' sætning

Et af de enkleste eksempler på dimensionsanalyse er det nedenstående bevis for Pythagoras' sætning. Udover at være enkelt, er eksemplet også dialogåbnende i forhold til umiddelbart dimensionsanalyse-skeptiske matematikere. I online-versionen af artiklen findes flere, mere komplicerede, men også mere repræsentative eksempler.

De fleste kender formlen $c^2 = a^2+b^2$, som udtrykker forholdet mellem sidelængderne i en retvinklet trekant. Det er en meget berømt geometrisk sætning, som der findes hundredvis af mere eller mindre komplicerede beviser for (se fx. www.cut-the-knot.org/pythagoras/ som indeholder en samling på 120 af slagsen). Her vil vi skitsere et dimensionsargument for sætningen.

En retvinklet trekant er entydigt bestemt ud fra længden af hypotenusen, c, og den mindste vinkel, θ. Arealet A af trekanten er således også en entydigt bestemt funktion af disse to størrelser.

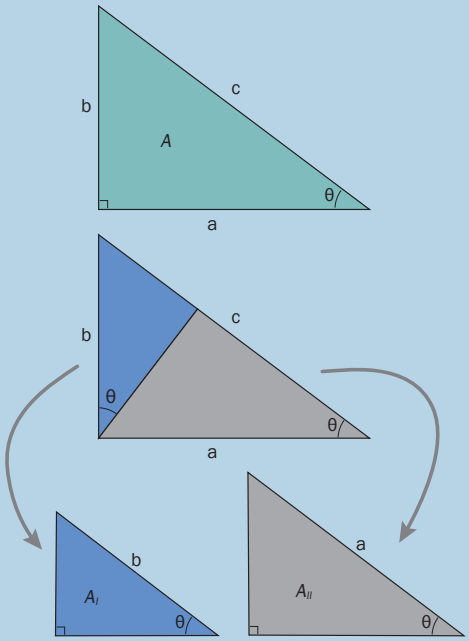
På figuren er den retvinklede trekant opdelt i to retvinklede trekanter med henholdsvis arealerne A_I og A_{II}. Det ses af figuren, at de to trekanter har samme form som den store trekant, begge med vinklen θ som deres mindste vinkel, men med hypotenuserne b og a.

Arealet af den oprindelige trekant opfylder:

$$A = A_I + A_{II}$$

Nu vil vi bruge dimensionsanalyse til at gå et skridt videre i beviset.

Da vinklen θ er dimensionsløs (den har ikke en fysisk enhed), og arealet af trekanten skal have dimensionen areal (længde gange længde), må A af dimensionsgrunde være tvunget til at have formen $c^2 \cdot f(\theta)$, hvor f er en entydig funktion alene af θ. Vi vil ikke på andre måder



kunne få dimensionen areal ud af størrelserne c og θ. Tilsvarende gælder $A_{II} = a^2 \cdot f(\theta)$ og $A_I = b^2 \cdot f(\theta)$.

Læg mærke til, at vi ikke behøver kende det eksakte udtryk for $f(\theta)$. Vi har kun brug for, at $f(\theta)$ er entydigt bestemt. (I dette tilfælde kan vi dog beregne $f(\theta) = 1/2 \cdot \cos(\theta) \cdot \sin(\theta)$.)

Sætter vi udtrykkene ind i $A = A_I + A_{II}$ får vi:

$$c^2 \cdot f(\theta) = a^2 \cdot f(\theta) + b^2 \cdot f(\theta)$$

Ved division med $f(\theta)$ på begge sider af lighedstegnet fås

$$c^2 = a^2 + b^2,$$

altså Pythagoras' sætning.

Ved at tælle har man i forhistorisk tid kunnet nå frem til, at mine otte får plus dine fem får tilsammen udgør en flok på tretten får. Ligeledes har man kunnet finde ud af, at hvis vi bunker mine otte mursten med dine fem mursten, så har vi tretten mursten i bunken. Abstraktionsspringet består da i, at otte plus fem er tretten, uanset om det drejer sig om får, mursten eller alt muligt andet. Med vor tids skrivemåde gælder udsagnet: 8 + 5 = 13, generelt for tal. At der er tale om en abstraktion ses af, at det er noget, der kræver undervisning i de små klassetrin i skolen. Når man i dag snakker om at tælle på fingrene har det altså ikke så meget med fingre at gøre, som det har med en abstrakt tænkemåde at gøre.

Det andet abstraktionsspring består i at erstatte tal med bogstaver. Abstraktionsgraden øges selvfølgelig, når vi lader bogstaver være pladsholdere for tal. Lad os som illustration se på udtrykket (a+b)(a-b). Ved at benytte regnereglerne for tal for a og b kan parenteserne ganges ud til $a^2 - ab + ba - b^2$. Da ab = ba fås derfor $(a+b)(a-b) = a^2-b^2$ som en alment gældende formel. Da vi har udført de samme operationer med a og b, som vi kunne gøre med for eksempel tallene 17 og 3½, eller alle mulige andre par af tal, ses den enorme økonomiske gevinst ved abstraktionsspringet fra at regne med tal (aritmetik) til at regne med bogstaver (algebra): Ved at regne med bogstaver kan vi på en gang finde formler som gælder for alle talsituationer. Fantastisk!

Det tredje abstraktionsspring er, når bogstaverne i formelregning forstås, ikke som pladsholdere for tal, men som pladsholdere for fysiske størrelser. Hvis vi siger, at Lise og Anna vejer lige meget, eller at Jan vejer dobbelt så meget som de to hver for sig, er det udsagn om fysiske størrelser, ikke om tal. Vi behøver ikke tal for at bekræfte udsagnene. Det første udsagn kan bekræftes ved at Lise og Anna sætter sig i hver sin ende af en vippe. Det andet ved at Lise og Anna sammen sidder i den ene ende af vippen og Jan i

Dimensionsanalyse – sådan gør man!

Arbejdet med denne artikel har været vanskeligt: Hvordan skrive en artikel om udledning af formler uden illustrerende brug af formler? I det væsentlige har vi undgået formlerne, da erfaringen er, at mange, også blandt læserne af dette blad, går uden om formler. Til læsere, som har fået lyst til at vide mere om, hvordan man faktisk gør, har vi udarbejdet fem eksempler på dimensionsanalyse, som du kan finde på AktuelNaturvidenskab.dk og i en udvidet onlineversion af artiklen.

I dette ekstramateriale har vi udfoldet de to eksempler nævnt i artiklen om at finde svingningstiden og omløbstiden for henholdsvis et pendul og en satellit, og hvorfor permeabiliteten nødvendigvis må afhænge af kvadratet på størrelsen af partiklerne, når vand strømmer gennem materialer (Darcys lov).

Desuden har vi et eksempel fra biologiens verden om, hvor hurtigt fugle basker med vingerne i stillestående luft samt to yderligere fysikeksempler om størrelsen på et atom, og hvor hurtigt sand løber gennem et timeglas på månen.

den anden. Ligningerne $m_L = m_A$ og $m_L+m_A = m_J$ for masserne af Lise, Anna og Jan giver mening, uden at vi skal en omvej omkring enheder og tal. I fysikken har det gennem de seneste par hundrede år gradvist stået klarere, at bogstavregningen i teoretisk fysik repræsenterer regning med fysiske størrelser fremfor med tal. Massen af Lise er ikke et tal. Den kan udtrykkes ved et tal gange en enhed, for eksempel 50 kg. Men den kan også beskrives som 50.000 g. Eller som 110,23 lb. Uanset hvilke enheder med dertil hørende tal, vi benytter, er det størrelsen massen af Lise, der er tale om. Og i teoretisk fysik behøver vi ikke vælge enheder for at foretage beregninger. Det er først, når vi sammenligner med målinger, at vi har behov for enheder og dertil hørende tal.

Størrelser og dimensioner i fysik

I geometri kan vi sige, at for eksempel hypotenusen i en retvinklet trekant har dimensionen længde. Man skriver [c] = L, som skal læses som "dimensionen af c er længde". Med længde som enkeltstående basisstørrelsesart gælder så $[c^2] = L^2$, idet c^2 er af den af basisstørrelsesarten længde afledte størrelsesart areal. Da størrelsesarten vinkel er afledt som forholdet imellem to længder, kan vi for en vinkel, φ, skrive [φ] = 1, hvor 1-tallet markerer, at størrelsesarten vinkel regnes dimensionsløs. Alt sammen meget formelt og umiddelbart værdiløst, hvis ikke det kunne fungere som en overgang til, hvordan der arbejdes med basisstørrelsesarter, afledte størrelsesarter og dertil svarende dimensioner i fysik.

Det internationalt vedtagne SI-system, som lærebøgerne i fysik nu om dage skrives i overensstemmelse med, er opbygget af to omgange. Først er det fastlagt, hvilke basisstørrelsesarter systemet skal bygges på. Nemlig størrelsesarterne længde, masse, tid, elektrisk strømstyrke, temperatur, lysstyrke og stofmængde. Derefter er der besluttet enheder for basisstørrelsesarterne, nemlig meter, kilogram, sekund, ampere, kelvin, candela og mol. Herudfra kan der defineres afledte størrelsesarter med hertil hørende dimensioner og afledte enheder. For eksempel er størrelsesarten hastighed afledt af basisstørrelsesarterne længde og tid med dimensionen L/T og enheden m/sek, idet dimensionen af størrelsesarten tid kaldes T. Kort fortalt har SI-systemet haft en dobbelt betydning for fysik. I forhold til eksperimentalfysik leverer det måleenheder. I forhold til teoretisk fysik styrer det, i kraft af valget af basisstørrelsesarter, måden formler skrives på, og måden størrelsesregning i fysik foregår på.

Teoretisk forskning ved hjælp af dimensionsanalyse

Når bogstavregning i fysik forstås som regning med fysiske størrelser med forskellige dimensioner, er det ikke så svært at forstå, hvad dimensionsanalyse bygger på. Det giver kun mening at addere, subtrahere eller sætte fysiske størrelser lig med hinanden, hvis de har samme dimension. Derimod giver det mening at gange og dividere fysiske størrelser med hinanden til nye slags størrelser. For eksempel kan hastigheden 1 m/sek opnås ved at dividere 18 meter med 18 sekunder. Ved bogstavregning med fysiske størrelser ganger og dividerer vi med bogstaverne som pladsholdere for pakker af tal gange enheder. Men vi kan kun addere og subtrahere bogstaver, der repræsenterer størrelser med samme dimension. Og vi kan kun sætte bogstavudtryk lig hinanden, hvis størrelserne på hver side af lighedstegnet har samme dimension. Det er kernen i dimensionsanalyse.

Ved anvendelse af Newtons mekanik kan svingningstiden for penduler udregnes til at være givet ved formlen $\tau = 2\pi\sqrt{l/g}$, hvor τ er pendulets svingningstid, og l og g er henholdsvis pendulets længde og tyngdefeltstyrken på Jorden. Ved en anden slags anvendelse af Newtons mekanik kan omløbstiden

for satellitter nær Jordens overflade udregnes til at være givet ved formlen $\tau = 2\pi\sqrt{R/g}$, hvor τ nu er omløbstiden for satellitter, og R og g henholdsvis er Jordens radius og tyngdefeltstyrken på Jorden.

De to formler, kan også udledes ved dimensionsanalyse – bortset fra faktoren 2π i formlerne. Ifølge formlerne afhænger hverken pendulsvingningstider eller satellit-omløbstider af henholdsvis pendulernes eller satellitternes masser. Det er umiddelbart til at forstå ud fra dimensionsanalyse. Vi kan nemlig ikke få dimensionerne til at passe, hvis vi i en formel for τ lader en masse indgå på højre side af formlen. Dimensionsanalysen forklarer også, hvorfor formlerne er ens bortset fra ombytningen af pendullængden og Jordens radius, selvom måderne de traditionelt udledes på afviger fra hinanden.

Et værktøj ikke kun for fysikere

Også uden for fysik kan man teoretisk tænke sig til formler ved hjælp af dimensionsanalyse. I hydrogeologi bruger man udtrykket permeabilitet til at beskrive vands bevægelse igennem materialer i undergrunden (for eksempel sand eller grus). Ifølge lærebogslitteraturen i hydrogeologi er det en måleerfaring, at permeabiliteten er proportional med kvadratet på størrelsen af partiklerne i materialet. Dimensionsanalyse viser imidlertid, at det, med de gjorte antagelser i hydrogeologi, ikke kan være anderledes. Når måleresultatet viser, at permeabiliteten afhænger af kvadratet på størrelsen af partiklerne, er det derfor ikke så interessant, som hvis det viste en afvigelse herfra. For så ville det nemlig rette søgelyset imod primære antagelser i hydrogeologien. Man kan også teoretisk tænke sig til resultater i biologi ved hjælp af dimensionsanalyse. Hvor hurtigt basker en stillestående lærke med vingerne, sammenlignet med hvor hurtigt en stillestående høg basker med sine vinger?

Dimensionsanalyse er altså ikke forbeholdt fysik – alle fag, der tænker i fysiske størrelser, kan have glæde af det. Permeabiliteten i hydrogeologi og vingebaskefrekvensen for stillestående fugle er lige såvel fysiske størrelser (med reference til SI-systemet), som de fysiske størrelser, vi udregner formler for i fysik.

Men opfattelsen af formler er afgørende for at udføre dimensionsanalyse både i fysik og andre fag.

Den mest udbredte opfattelse af formler er, at de er en form for stenografisk fremstilling af måleresultater. Og opfattet således, giver det ikke mening at nå frem til formlerne ved hjælp af dimensionsanalyse. Vi må da nøjes med enhedskontrol.

Dimensionsanalyse hænger sammen med en teoretisk tilgang til udledningen af formler: Givet de og de formler i forvejen sammenholdt med de og de antagelser til lejligheden, hvad fører det til? Det er i en sådan situation, at dimensionsanalysen kan være teoretisk vejledende.

At dimensionsanalysen er et hemmeligt våben for fysikere frem for udøvere af andre fag, hænger nærliggende sammen med tyngden af formaliseret teori i fysik. Ikke desto mindre er dimensionsanalysen overset som tænkemåde, ikke alene i andre fag, men også blandt mange fysikere . ■

Videre læsning:
Jan de Boer, Symboler og betegnelser i matematikken og fysikken, Fysisk Tidskrift 86 (1988) 49

Einstein gjorde noget tilsvarende vores bevis her for Pythagoras sætning som 11-årig, se S. Strogatz, Einsteins first proof, The New Yorker (19 november 2015).

For yderligere inspiration fra vores hånd se J.H. Jensen, Atlanterhavsbølger, KVANT December 2015, 29, J. H. Jensen, Bohrs atommodel, KVANT Maj 2016, 33 og T. Hecksher, Insights through dimensions, Nature Physics, 13 (2017), 1026.

En mere omfattende introduktion på dansk til dimensionsanalyse kan findes ved at google IMFUFA tekster og gå til tekst nr. 269, en studenterprojektrapport fra RUC skrevet af Tine Guldager Christiansen, Ken Andersen, Nikolaj Hermann og Jannik Rasmussen i 1994.

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Fysikernes hemmelige våben

Fem eksempler på dimensionsanalyse

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Eksempel 1: Pendul og satellit

Vi vil finde formler for svingningstiden for et pendul og omløbstiden for en lavt hængende satellit ved hjælp af dimensionsanalyse.

Lad os først finde en formel for svingningstiden for et pendul. Et simpelt pendul er en svingende sten bundet til en snor, der er fixeret i dens anden ende. Hvad afhænger svingningstiden af? Længden af pendulsnoren, l, må spille en rolle. Erfaringer med for eksempel pendul-ure er, at svingningstiden vokser med længden af deres pendulstænger. Tyngdefeltstyrken ("tyngdeacce-lerationen"), g, på Jordens overflade må også være medbestemmende for svingningstiden. På Månen forventer vi for eksempel, at et givet pendul vil have en anden svingningstid end på Jorden. Måske afhænger svingningstiden også af pendulstenens masse, m?

Da indbyrdes additioner og subtraktioner af l, g og m ikke giver mening på grund af deres forskellige dimensioner, som er L, LT^{-2} og M, forsøger vi os med

$$\tau = \text{tal} \times l^{\alpha} g^{\beta} m^{\gamma} \,, \tag{1}$$

som formel for svingningstiden τ, hvor α, β og γ i første omgang er ubekendte. I anden omgang må vi imidlertid kræve, at de to sider af ligning (1) har samme dimension. Og det ses de kun at have, hvis

$$T = L^{\alpha}(LT^{-2})^{\beta} M^{\gamma} = L^{\alpha+\beta} T^{-2\beta} M^{\gamma} \,. \tag{2}$$

Da basisdimensionerne L, T og M per definition ikke kan reduceres til hinanden, skal potenserne i ligning (2) være de samme på hver side af ligningen for hver basisdimension for sig. Det betyder, at de tre ubekendte α, β og γ skal være løsning til de tre ligninger:

$$L: \quad 0 = \alpha + \beta \tag{3}$$

$$T: \quad 1 = -2\beta \tag{4}$$

$$M: \quad 0 = \gamma \,, \tag{5}$$

idet vi på venstre side af ligning 2 i stedet for T kunne skrive $L^0T^1M^0$, og en dimension opløftet til potensen 0 regnes for det dimensionsløse tal 1.

Ligning (5), $\gamma = 0$, viser, at svingningstiden, ifølge ligning (1), ikke afhænger af m. Vi havde altså ikke behøvet medtage stenens masse som styrende parameter i starten af vores dimensionsanalyse. På den anden side er uafhængigheden af pendulmassen altså et resultat, der fremkommer tvingende af analysen.

Af ligning (4) fås $\beta = -1/2$, som indsat i ligning (3) giver $\alpha = 1/2$. Ved indsætning af disse værdier for α og β i ligning (1), når vi herefter frem til, at formlen for svingningstiden er:

$$\tau = \text{tal} \times (l/g)^{1/2} \tag{6}$$

En normal matematisk/fysisk udledning fører til en formel magen til. Det ekstra, der opnås ved ikke at skyde genvej ved hjælp af dimensionsanalyse, er, at tallet i ligning (6) kan findes til at være 2π.

(Faktisk afhænger svingningstiden af et pendul ved store udsving, udover af pendulsnorens længde og tyngdefeltstyrken, også af udsvingets størrelse. Så tallet i ligning (6) burde strengt taget rettes til at være en ukendt funktion af den maksimale udslagsvinkel. Vores dimensionsanalyse udledning af ligning (6) er kun rigtig for små penduludsving.)

Lad os dernæst finde omløbstiden for en lavtflyvende satellit. Hvad afhænger den af? Vi kan igen tænke på tyngdefeltstyrken ved Jordens overflade, g, og satellittens masse, m, i stedet for pendulstenens masse. Herudover må, modsvarende pendulsnorens længde, Jordens radius, R, være en bestemmende parameter for satellittens omløbstid. Nået så langt, viser det sig, at dimensionsanalysen til udledning af en formel for satellittens omløbstid er helt magen til udledningen af formlen for pendulets svingningstid. I begge tilfælde ønsker vi at finde en karakteristisk tid som funktion af en karakteristisk længde, massen af det, der bevæger sig, og tyngdefeltstyrken. Dimensionsanalyserne forløber derfor ens i de to tilfælde. Hvad angår satellittens omløbstid, er svaret derfor, at den ikke afhænger af satellittens masse, og er bestemt af ligning (6) med R indsat i stedet for l. Ved en normal matematisk/fysisk udledning findes tallet i formel (6) også i dette tilfælde at være 2π.

De normale matematisk/fysiske udledninger af formlerne for pendulers svingningstider og lavtflyvende satellitters omløbstider er ikke analoge. På sin vis viser udledningerne ved hjælp af dimensionsanalyse derfor en dybere sammenhæng mellem de to fænomener, end det umiddelbart fremgår af de sædvanlige udregninger.

Eksempel 2: Fuglevingebaskning

Hvor hurtigt basker en fugl, der holder sig stille i luften over et sted i landskabet i vindstille vejr? Vores erfaring siger os, at små fugle basker hurtigere med vingerne end store fugle. Men kan vi komme det lidt nærmere og sige noget mere kvantitativt om sammenhængen?

Lad os analysere situationen lidt. Hvis fuglen holder sig svævende og altså hverken taber eller øger sin højde, må fuglens vinger skubbe præcis nok luft nedad pr tid til at balancere tyngdekraften, F_g. Vi kan kalde effekten af fuglens basken med vingerne for en opdriftskraft, F_o, og for at fuglen kan holde sig stille i luften over det samme punkt i landskabet skal disse kræfter altså balancere, $F_g = F_o$.

Tyngdekraften kender vi, $F_g = mg$, hvor m er fuglens masse og g er tyngdefeltsstyrken. Men hvad kan opdriftskraften afhænge af? Den afhænger nok af vingebaskningsfrekvensen, f, altså hvor mange gange i sekundet vingerne bevæges op og ned. Derudover må opdriften også afhænge af luftens densitet, ρ: jo tyndere luft, jo mere skal der skubbes nedad for at balancere tyngdekraften. Desuden må størrelsen af vingerne også betyde noget for, hvor meget luft, der flyttes. Hvis vi antager, at fuglevinger er nogenlunde ensdannede, kan vi beskrive størrelsen af vingerne ved fx. deres længde, altså vingefanget, s. Hvis vi også antager at måden forskellige fugle basker med vingerne er ens, når vi frem til at opdriftskraften må være bestemt af f, ρ og s:

$$F_o = \text{tal} \times f^\alpha \rho^\beta s^\gamma \tag{1}$$

Dimensionen af frekvens er T^{-1}, dimensionen af densitet er masse pr volumen, altså ML^{-3}, og dimensionen af vingefanget er længde, L. Kravet om, at begge sider af ligning (1) har samme dimension (nemlig kraft, $[F] = MLT^{-1}$), betyder at eksponenterne α, β og γ skal matches med eksponenterne på venstresiden, altså

$$M\,L\,T^{-2} = (T^{-1})^\alpha \cdot (ML^{-3})^\beta \cdot (L)^\gamma = T^{-\alpha} M^\beta L^{-3\beta} L^\gamma = M^\beta\, L^{-3\beta+\gamma}\, T^{-\alpha}\,, \tag{2}$$

hvilket leder til en ligning for eksponenterne for hver dimension:

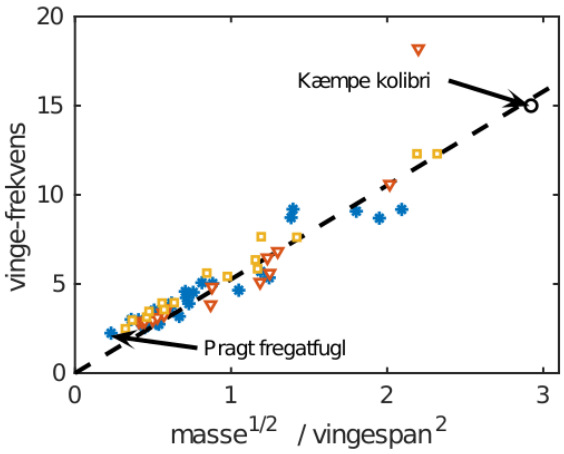
$$T: \quad -2 = -\alpha \tag{3}$$
$$M: \quad 1 = \beta \tag{4}$$
$$L: \quad 1 = -3\beta + \gamma \tag{5}$$

Vi har altså $\alpha = 2$ (ligning (3)) og $\beta = 1$ (ligning (4)), som indsat i den sidste ligning (5) giver $\gamma = 4$. Af dimensionsgrunde må der derfor for F_o nødvendigvis gælde

$$F_o = \text{tal} \times f^2 \rho s^4\,. \tag{6}$$

Da opdriftskraften F_o skal balancere tyngdekraften F_g har vi derfor

$$\text{tal} \times f^2 \rho s^4 = mg \tag{7}$$



Figur 0.1 Lineariseret plot af vingefrekvens data.

Her kan vi isolere f og dermed svare på vores oprindelige spørgsmål: hvor hurtigt basker fugle med vingerne? Dimensionsanalysesvaret er således $f^2 = \text{tal} \times \frac{mg}{\rho s^4}$ og dermed fås at

$$f = \text{tal} \times \left(\frac{mg}{\rho s^4}\right)^{1/2} = \text{tal} \times \left(\frac{g}{\rho}\right)^{1/2} \frac{m^{1/2}}{s^2} \,. \tag{8}$$

Da densiteten af luft ρ og tyngdefeltsstyrken g er den samme for forskellige fuglearter er forudsigelsen altså at vingefrekvensen er proportional med kvadratroden af massen over vingespan i anden, $f \propto m^{1/2}/s^2$. Forventningen er at en graf der viser vingefrekvensen f plottet som funktion af $\sqrt{m}/s^2$ giver en ret linie.

Men passer denne formel så? Faktisk findes der data i litteraturen på fugles vingefrekvens, deres masse og vingefang, som vi kan teste vores formel på[1]. Figur 0.1 viser tre datasæt (blå, orange og gul) som følger forudsigelsen af proportionalitet mellem vingefrekvens og $m^{1/2}/s^2$. Vores forudsigelse virker altså rimelig godt! Hældningen af plottet kan bruges til at bestemme den ukendte talfaktor som dimensionsanalysen ikke kan sige noget om.

Data er fra følgende referencer:

C. J. Pennycuick "Predicting wingbeat frequency and wavelength of birds", J. Exp. Biol. **150**, p. 171-185 (1990)
C. J. Pennycuick "Wingbeat frequency of birds in steady cruising flight: new data and improved predictions", J. Exp. Biol. **199**, p. 1613-1618 (1996)
C. J. Pennycuick "Speeds and wingbeat frequencies of migrating birds compared with calculated benchmarks", J. Exp. Biol. **204**, p. 3283-3294 (2001)

[1] Det er dog fugle i flugt og ikke stillestående fugle, men hvis det antages, at deres vingefrekvens ikke er væsentlig forskellig i de to situationer, kan vi teste vores formel på disse data

Eksempel 3: Vandstrømning gennem sand

I hydrogeologi optræder den såkaldte Darcy's lov. Den udtrykker, at strømningshastigheden, v, af en væske gennem et porøst materiale, f.eks. sand, antages proportional med trykfaldet per længdeenhed, $-dP/dx$, hvor proportionalitetskonstanten, k, kaldes for permeabiliteten af den pågældende væske gennem det pågældende materiale. Altså, Darcy's lov:

$$v = -k dP/dx \tag{1}$$

Ifølge lærebogslitteraturen i hydrogeologi er det en måleerfaring, at permeabiliteten er proportional med kvadratet på kornstørrelsen af kornene, som materialet består af. Men det behøver vi imidlertid ikke måle os til. Det kan ikke være anderledes af dimensionsgrunde.

Hvad kan k afhænge af? Permeabiliteten må afhænge af, hvilken væske der er tale om. Det må være sværere at presse mere sejtflydende væsker gennem porerne i materialet end mindre sejtflydende. Vi antager derfor, at k afhænger af η, væskens viskositet, som er et mål for, hvor sejlflydende den er. Vi kunne måske også forestille os, at k afhænger af væskens massefylde, ρ. Herudover må permeabiliteten afhænge af størrelser, former og sammenpakninger i materialet, hvor en given fordeling af kornstørrelser, kornformer og deres sammenpakningsmønstre antages at kunne beskrives ved en middelkornstørrelse, d, og en dimensionsløs funktion, $F(f)$, af en række geometriske forhold f karakteristisk for den pågældende fordeling af størrelser, former og sammenpakninger.

Hvis vi antager, at der ikke er yderligere forhold, der påvirker permeabiliteten, må en formel for den søges blandt:

$$k = \eta^{\alpha}\, \rho^{\beta}\, d^{\gamma}\, F(f)\,, \tag{2}$$

hvor α, β og γ skal vælges således, at højre side af formlen får dimension magen til k og en eventuel ukendt talfaktor er indlemmet i $F(f)$.

Ifølge definitionen af k i ligning 1 ses dens dimension at være:

$$[k] = [v] \cdot [dx]/[dP] = (LT^{-1}) \cdot L/(ML^{-1}T^{-2}) = M^{-1}L^{3}T\,, \tag{3}$$

idet dimensionen af tryk (og trykændring, dP) er $ML^{-1}T^{-2}$, dimensionen af længde (og længdeændring, dx) er L, og dimensionen af hastighed, v, er LT^{-1}.

Da $[\eta] = ML^{-1}T^{-1}$, $[\rho] = ML^{-3}$ og $[d] = L$, skal α, β og γ, for at sikre samme dimension på begge sider af lighedstegnet i ligning (2), derfor opfylde ligningen:

$$M^{-1}L^{3}T = \left(M\,L^{-1}\,T^{-1}\right)^{\alpha} \cdot \left(M\,L^{-3}\right)^{\beta} \cdot L^{\gamma}) = M^{\alpha+\beta}\,L^{-\alpha-3\beta+\gamma}\,T^{-\alpha}\,. \tag{4}$$

Da basisdimensionerne M, L, og T per definition ikke kan udtrykkes ved hinanden, må potenserne svarende til hver basisdimension være den samme på begge sider af lighedstegnet.

$$M: \quad -1 = \alpha + \beta \tag{5}$$
$$L: \quad 3 = -\alpha - 3\beta + \gamma \tag{6}$$
$$T: \quad 1 = -\alpha \tag{7}$$

Ligning (7) giver $\alpha = -1$ direkte. Dette indsat i ligning (5) giver $\beta = 0$. Med disse to resultater kan det ses at ligning (6) kun opfyldes, hvis $\gamma = 2$.

Hvis Darcy's lov kan gøres gældende, er permeabilitetetskonstanten i den derfor af dimensionsgrunde tvingende givet ved et udtryk af formen:

$$k = \eta^{-1} d^2 F(f) \tag{8}$$

Her er den dimensionsløse funktion $F(f)$ ukendt. Men sikkert er det, at k ikke kan afhænge af massefylden af den gennemstrømmende væske, og at den kun kan afhænge kvadratisk af d.

Eksempel 4: Hvor stort er et atom?

Hvis man spørger en fysiker vil svaret være 1 Å, dvs. 10^{-10} m – det er noget enhver fysiker ved. Men kan vi regne det ud? Og i særdeleshed: kan vi regne det ud ved hjælp af dimensionsanalyse? Det (måske) overraskende svar er ja. Og ikke nok med det – det var faktisk sådan selveste Niels Bohr blev inspireret til sin berømte atommodel i kvantemekanikkens tidlige dage.

Lidt historisk baggrund

I starten af forrige århundrede vidste man godt, at et atom består af ladede partikler. I 1911 viste Rutherford, at atomet mest består af tomt rum med en meget lille positivt ladet kerne og negative ladninger kredsende omkring, som planeterne omkring solen. I sin artikel fra 1913 argumenterer Niels Bohr – helt korrekt – for, at den klassiske fysik ikke kan forklare atomets stabilitet: Kredsende ladninger er konstant accelererede og burde derfor ifølge elektrodynamikken udsende elektromagnetisk stråling, tabe kinetisk energi og dermed spiralere ind mod kernen for til sidst at blive opslugt. Intet af dette stemmer med de observationer, man har for atomer. Atomer kan godt nok udsende stråling; dog ikke et kontinuert spektrum som forudsagt af elektrodynamikken, men i diskrete spektrale linjer. Og atomer kollapser ikke. Ved at antage, at energien på atomart niveau er kvantiseret og at elektronerne kun kan befinde sig i bestemte baner omkring kernen, udledte Niels Bohr sin berømte formel for spektrallinjerne af hydrogenatomet. Og han bemærkede, at ved at indføre Plancks konstant h kan man udlede den rigtige størrelse af atomet ved dimensionsanalyse.

Dimensionsanalyse

Hvis vi skal give et estimat af størrelsen af hydrogenatomet baseret på grundlæggende fysiske konstanter, skal vi først identificere de relevante konstanter for problemet. Da vi betragter et system bestående af en elektron, der kredser om den meget tungere proton, vil de oplagte størrelser at inddrage være elementarladningen e, elektronens masse m_e og vakuumpermitiviteten ε_0. Og så altså Plancks konstant, h. Dimensionen af disse størrelser er[1]:

elementarladningen:	$[e] = Q$
elektronens masse:	$[m_e] = M$
permitivitetskonstanten:	$[\varepsilon_0] = M^{-1}L^{-3}T^2Q^2$
Plancks konstant:	$[h] = ML^2T^{-1}$

Som et forsøg på at finde et udtryk for størrelsen af brintatomet, Bohr radius a_{Bohr}, ganger vi

[1] Her bruger vi grundstørrelsesarterne: masse M, længde L, tid T og ladning Q. SI-systemet opererer ikke med ladning men med strømstyrke som grundstørrelsesart. I dimensionsanalyse kan man selv vælge hvilke grundstørrelsesarter man vil arbejde med, så længe man er konsekvent.

potenser af e, m_e, ε_0 og h sammen. Kan man finde værdier for α, β, γ og δ således at formlen

$$a_{\text{Bohr}} = e^{\alpha} m_e^{\beta} \varepsilon_0^{\gamma} h^{\delta} \tag{1}$$

respekterer kravet om ens dimension på begge sider af lighedstegnet? Kravet fører til følgende ligning for dimensionerne

$$L = Q^{\alpha} M^{\beta} (M^{-1} L^{-3} T^2 Q^2)^{\gamma} (M L^2 T^{-1})^{\delta} = Q^{\alpha+2\gamma}\, M^{\beta-\gamma+\delta}\, T^{2\gamma-\delta}\, L^{-3\gamma+2\delta}\,, \tag{2}$$

og da grundstørrelsesarterne, M, L, T og Q, pr definition ikke kan udtrykkes ved hinanden skal, potenserne på begge sider af lighedstegnet for hver af grundstørrelsesarterne altså være den samme:

$$Q: \quad 0 = \alpha + 2\gamma \tag{3}$$
$$M: \quad 0 = \beta - \gamma + \delta \tag{4}$$
$$T: \quad 0 = 2\gamma - \delta \tag{5}$$
$$L: \quad 1 = -3\gamma + 2\delta \tag{6}$$

Vi får således fire ligninger med fire ubekendte. Ud fra ligning (5) og (6) findes $\gamma = 1$ og $\delta = 2$, som indsat i ligning (3) og (4) giver $\alpha = -2$ og $\beta = -1$. Vi kan altså godt finde værdier for α, β, γ og δ, der respekterer kravet om ens dimension på begge sider af lighedstegnet af ligning (1). Og ikke nok med det: dimensionsanalysen fører til, at enhver teori for brintatomet baseret på konstanterne e, m_e, ε_0 og h af dimensionsgrunde nødvendigvis må føre til resultatet

$$a_{\text{Bohr}} = \text{tal} \times \frac{h^2 \varepsilon_0}{m_e e^2}\,. \tag{7}$$

Under antagelse af, at talfaktoren er af størrelsesorden 1[2] findes ved indsættelse af de numeriske værdier af de indgående konstanter

$$a_{\text{Bohr}} \approx \frac{(6.626 \cdot 10^{-34}\ \text{Js})^2\, 8.854 \cdot 10^{-12}\ \text{F/m}}{9.109 \cdot 10^{-19}\ \text{kg}\,(1.602 \cdot 10^{-19}\ \text{C})^2} = 1.66 \cdot 10^{-10} m = 1.66\ \text{Å} \tag{8}$$

hvilket netop – som Niels Bohr argumenterede – cirka er størrelsen af brintatomet.

Referencer:

N. Bohr "On the constitution of atoms and molecules", Phil. Mag. **26**, 1-25 (1913)
N. Bohr "Om Brintspektret", Fysisk Tidsskrift, **12**, (1913)

[2] Som SI systemet er standardiseret optræder der sjældent talfaktorer i fysiske formler, der er en størrelsesorden større eller en størrelsesorden mindre end 1. De mest almindelige talfaktorer skyldes geometri og indeholder π (jævnfør eksempel 1 om pendul og satellit). På grund af ukendskabet til talfaktoren leverer dimensionsanalysen ikke resultatet 1.66 Å. Det er mere rigtigt at sige, at dimensionsanalysen viser, at Bohr radius ligger mellem 10^{-11} og 10^{-9} m, hvilket – når man regner med størrelses ordener – er i pæn overensstemmelse med 10^{-10} m.

Eksempel 5: Hvor lang tid er sandet om at løbe gennem et timeglas på månen?

Det er måske et lidt fjollet spørgsmål at stille, for det bliver næppe et timeglas, man bruger til at måle tiden på månen. Alligevel er det et godt eksempel på, hvordan man ved hjælp af dimensionanalyse ret let kan give et bud på svar, men til gengæld hurtigt ender i noget virkelig indviklet, hvis man skal stille ligninger op for problemet og løse dem.

Hvad afhænger måden sandet løber igennem et timeglas så af? Tyngdefeltsstyrken, g, naturligvis. Og størrelsen af hullet, r, sandet skal løbe igennem.

Men den afhænger også af gnidningskoefficienten mellem sandkorn og glas, μ_g, gnidningskoefficienterne mellem sandkornene indbyrdes, μ_s, størrelsen af sandkorn, sandkornsformerne, fordelingen af sandskornsstørrelser og -former og timeglassets form. Imidlertid kan alle former og fordelinger beskrives dimensionsløst ved vinkler $\theta_1, \theta_2, \theta_3, \ldots$ og længdeforhold i forhold til størrelsen af hullet $l_1/r, l_2/r, l_3/r, \ldots$. Da også gnidningskoefficienter er dimensionsløse ender vi op med at antage, at gennemløbstiden, τ, må være givet ved en formel af udseendet

$$\tau = F(\mu_g, \mu_s, \theta_1, \theta_2, \theta_3, \ldots, l_1/r, l_2/r, l_3/r, \ldots)\, r^\alpha g^\beta\,, \qquad (1)$$

idet det kun er størrelserne r og g, der bidrager til at give dimensionen tid. I funktionsudtrykket for F indgår kun dimensionsløse størrelser, og F er således selv dimensionsløs. Dimensionerne af r og g er $[r] = L$ og $[g] = LT^{-2}$.

Vi ønsker at finde en tid ud fra disse størrelser, hvilket leder til følgende ligning for dimensionerne

$$T = L^\alpha (LT^{-2})^\beta. \qquad (2)$$

For at få dimensionerne til at passe skal der for potenserne α og β gælde

$$T: \quad -2\beta = 1 \qquad (3)$$
$$L: \quad \alpha + \beta = 0\,. \qquad (4)$$

Ligning (3) giver $\beta = -1/2$, hvilket indsat i ligning (4) giver $\alpha = 1/2$.

Tiden τ, det tager sandet at løbe gennem timeglasset, er derfor givet ved

$$\tau = F r^{1/2} g^{-1/2} = F\sqrt{\frac{r}{g}} \qquad (5)$$

Dimensionsanalysen giver et svar – bortset fra, at vi ikke kender den dimensionsløse funktion F. Funktionen F er den samme på jorden og på månen, men hvis vi derudover antager, at

alle argumenterne (dvs. gnidningskoefficienter, vinkler, fordelinger, osv.) også er de samme, vil funktions*værdien* være den samme på jorden og månen og derfor kan svaret alt andet lige end g skrives i termer af den tid, det tager på jorden (som jo netop er en time):

$$\begin{aligned} \frac{\tau_{\text{måne}}}{\tau_{\text{jord}}} &= \frac{F\sqrt{\frac{r}{g_{\text{måne}}}}}{F\sqrt{\frac{r}{g_{\text{jord}}}}} = \sqrt{\frac{g_{\text{jord}}}{g_{\text{måne}}}}, \text{ dvs.} \\ \tau_{\text{måne}} &= \sqrt{\frac{g_{\text{jord}}}{g_{\text{måne}}}}\tau_{\text{jord}} \end{aligned} \tag{6}$$

Ved indsættelse af $g_{\text{måne}} = 1.62$ m/s^2 og $g_{\text{jord}} = 9.82$ m/s^2 får vi, at det ville tage ca. 2 1/2 time for sandet at løbe igennem timeglasset på månen.

Så vidt dimensionsanalysens svar på spørgsmålet.

På Roskilde Universitet førte spørgsmålet – og dimensionssvaret – til et semesterprojekt. Studentergruppen satte sig for at undersøge problemet, dels med computersimuleringer, dels eksperimentelt. Metoden, de brugte til at variere tyngdefeltsstyrken, er den samme, som man bruger til at træne astronauter: de lavede en centrifuge til timeglasset og overvågede, hvordan sandet løb gennem timeglasset ved forskellige omdrejningshastigheder. Deres undersøgelser viste, at proportionaliteten $t \propto g^{-1/2}$ ikke holdt i deres forsøg. De fandt i stedet, at en anden potens gav en god beskrivelse af deres resultater, nemlig $t \propto g^{-3/4}$. Hvordan kan det nu være? Djævelen ligger i antagelsen *alt andet lige*. Projektet viste, at den antagelse ikke var rigtig, altså at funktionen f ikke har samme værdi ved forskellig g. Da sand ikke er en væske, der bare flyder, men består af større partikler af forskellige former, betyder pakningen (altså hvordan partiklerne ligger i forhold til hinanden) noget. Og pakningen afhænger tilsyneladende af tyngdefeltet. Pakning og flow af granulære materialer (som sand) er et forskningsområde i sig selv, og derfor er spørgsmålet om timeglasset på månen slet ikke så fjollet, som det umiddelbart lyder.

Videre læsning:

Filip Samuelsen, Helena Veldt, Peter Johannsen "The flow rate of granular material in an hourglass under various gravitational acceleration", NatBach 2. semester rapport, RUC
P. G. Hofmeister, J. Blum, and D. Heisselmann "The Flow Of Granular Matter Under Reduced-Gravity Conditions", **1145**, 71 (2009)

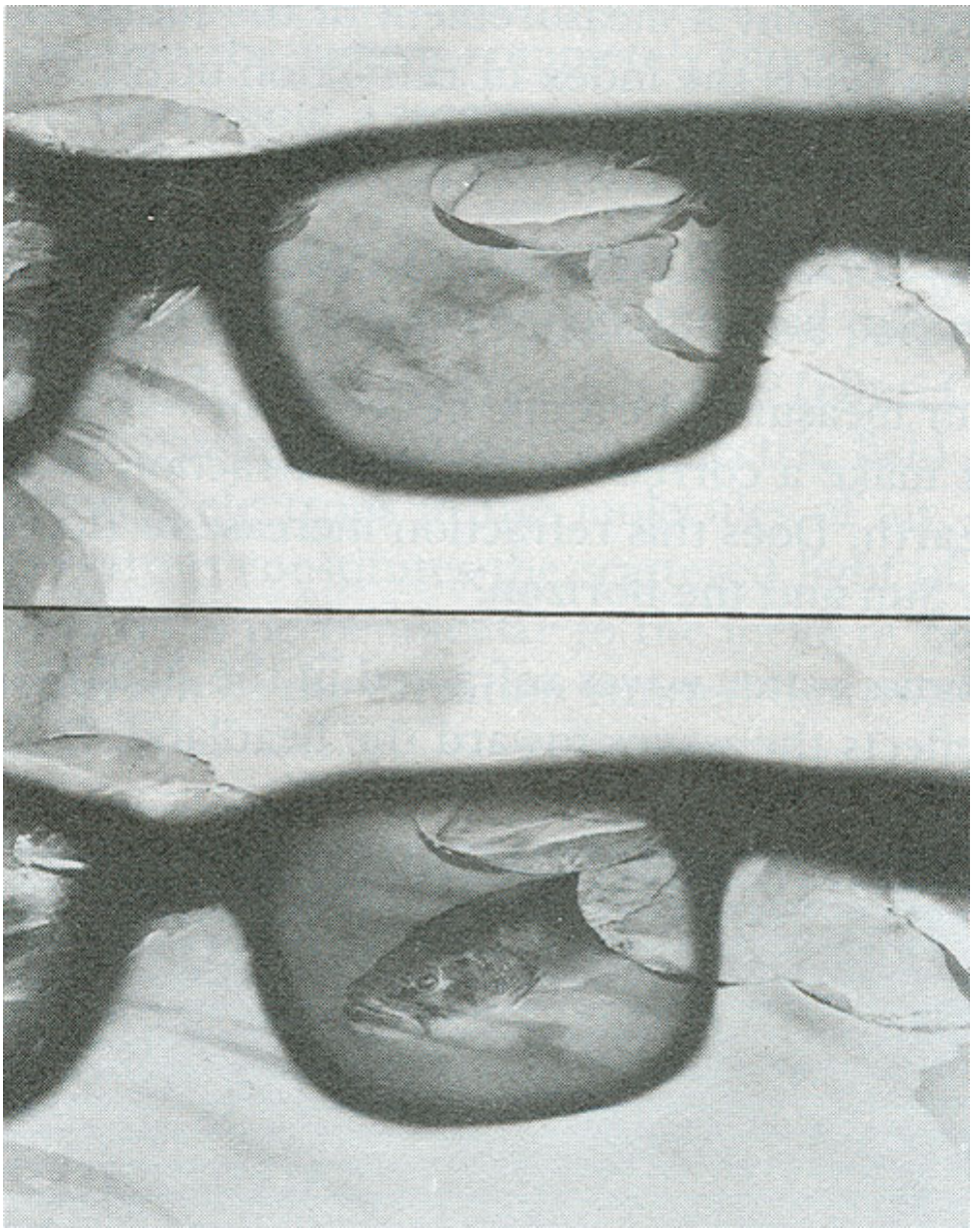


Fig. 37.32 When viewed through ordinary sunglasses (top), the fish is almost completely hidden by the glare reflected by the water. When viewed through Polaroid sunglasses (bottom), the fish becomes visible because the glare is reduced.

“At se er at se bort fra”

Den følgende artikel er fra *Mathematics as a Bridge Between the Disciplines, Procedings of* MACAS-2017 *Symposium (2018)*, svarende til et foredrag, jeg holdt på symposiet på DPU 27-29 Juni 2017. Foredraget blev filmet og kan findes på YOU-TUBE. Artiklen og foredraget er en udvidet oversættelse til engelsk af Matematiske modeller-vejledning eller vildledning? II, *IMFUFA tekst* nr. 199 (1990), s. 49-59.

MATHEMATICAL MODELLING – HIDING OR GUIDING?

Jens Højgaard Jensen[1]

Abstract

The increasing scientific management of technology and society, supported by increasingly powerful information technologies (IT), has lead, and leads to increasingly widespread use of mathematical models. This development gives rice to a democratic problem: How can ordinary people judge the conclusions delivered by mathematical models? Are the conclusions to be believed, since "mathematics do-not lye"? Or is it better to lean on the saying: "there are lies, damned lies and statistics"?

In the paper, I will illustrate a crude distinction between mathematical models derived from theories, and ad hoc mathematical models without reference to more global theories.

The distinction cannot be used to evaluate, whether specific models are hiding or guiding. Theory-derived models may be too idealized to be trusted. And ad hoc models may be trustworthy due to their richness of input data.

The value of the distinction is that it makes it clear that some mathematical models, the theory-derived models, besides of the possibility of evaluating them by comparing with empirical data, also can be evaluated by theoretical considerations. Evaluating ad hoc models are, in contrast, restricted to be done by empirical control only.

Thus, the distinction between theory-derived models and ad-hoc models may help ordinary people, not to distinguish between trustworthy and non-trustworthy models but to distinguish between the different qualities of the evaluation processes behind different sorts of models.

Introduction

This paper is not based on a concluded systematic empirical research. It is based on 40 years of unsystematically experiences with, and interests in, its subject. It may be called a philosophical essay.

In 1980 my interest in the problem, "Mathematical models – hiding or guiding?", lead to my guidance of a project group at Roskilde University on fourth semester,

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working half of their time on the problem "Matematiske modeller og videnskabsteoretiske problemer ved modeldannelse." ("Mathematical models and epistemological problems when modelling.") The group compared a model of fish populations, a hydrological model and a model of the geostationary satellite orbit around the earth epistemologically. They (we) concluded that the qualities of the evaluation processes behind the three models were very different. The theory-derived satellite model, besides of the possibility of evaluating it by comparing with empirical data, could also be evaluated by theoretical considerations. In contrast the ad hoc fish population model was only open for empirical evaluation without having a theory to be derived from. The satellite model and the fish population model represented the extremes in a span from theory-derived models to ad hoc models, whereas the studied hydrological model was a mixture of empirical grounded ad hoc assumptions and theoretical derivations.[2]

Frequently mathematical models are of the mixed type like the hydrological model. Nevertheless, more clear examples of theory-derived models, like the satellite model, and, in contrast to that, clear examples of ad hoc models like the fishing population model, may be needed as a pedagogical remedy to help the general public to judge whether apparent mathematical models are hiding or guiding. However, it is not sufficient to meet the needs of most people that the examples, like the satellite model and the fish population model, are clear. Besides being clear the examples should also be simpler than those models, if I, as I wanted, should contribute more to oblige mathematical models being a democratic problem: How can the general public judge the conclusions delivered by mathematical models? Are the conclusions to be believed, since "mathematics do-not lye"? Alternatively, is it better to lean on the saying: "There are lies, damned lies, and statistics?". Therefore, for 10 years I was searching simultaneously clear and simple examples illustrating the points already reached in 1980. The paper here is mainly the result of this search: As simple a presentation as possible of the reflections about "mathematical modelling – hiding or guiding?". It took half a year to develop the reflections and 10 years to find simple examples communicating the reflections.[3]

From 1987 to 2008 I and an IMFUFA colleague of mine, Bernhelm Boss-Bavnbek, organized 20 all day seminars at IMFUFA concerning the different use of mathematical models in different subjects and areas, e.g., statistics, applied

2 The project report of the students (in Danish) is included in *IMFUFA tekst* no. 26 (1980), Roskilde: Roskilde University. Also included is a more focused journal article (in Danish) by me.

3 The second article (in Danish) presenting the same points with revised examples is included in *IMFUFA tekst* no. 199 (1990), pp. 49-59

mathematics, physics, chemistry, biology, geography, computer science, economy, air pollution, meteorology, bridge building, climate, macroeconomics forecasting, nuclear power risks. This row of seminars gave me a lot of empirical experiences about the use of mathematical models. However, my points from 1980 were not affected. In what follows, I, for pedagogical reasons, present the points schematically by 6 theorems and 3 conclusions.

However, before presenting theorem 1, I want to remind you that the examples in my presentation are pedagogically chosen because they are simple and clear at the same time, not because of their democratic relevance. Hopefully, my discussion can help judging more important, though not simple, examples of mathematical modelling, like:

- The GPS-system
- Climate models
- Macroeconomic forecasting

I will return to and comment on these three examples of mathematical modelling in society in the end of the paper.

Theorem 1: Mathematical models can-not be refused

Mathematics do-not lie. Look at the crane in figure 1.

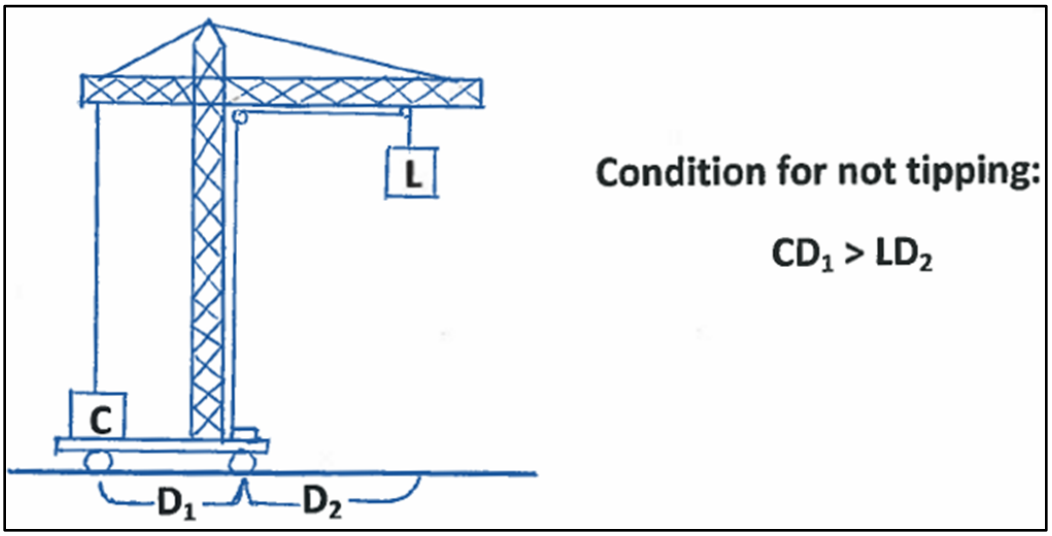


Figure 1: Crane width load L, counter weight C and distances from front wheels, D2 and D1, to L and C, respectively.

A crude estimate of the condition ensuring that the crane does not tip around the front wheels is $CD_1 > LD_2$. According to Newtonian mechanics the torque around the front wheels of the crane from the counter weight, have to be larger than the torque around the front wheels from the load. The estimate is crude because it assumes the crane construction to have infinitely small mass compared to L and

D. However, applying the same Newtonian mechanical torque rules as used to make the crude condition, a construction engineer can include the mass distribution of the crane in an almost exact calculation. The point is that the condition for not tipping can be found using mathematics in the framework of Newtonian mechanics. Therefore, both the construction workers on and around the crane, wanting to minimize risks of accidents, and the contractors, wanting to save money, can trust the calculation in spite of conflicting interests. The risks of being exposed when twisting the calculations, is big. Thus, the example proves that mathematical modelling is guiding and do-not lie.

Theorem 2: Mathematical models can-not be trusted

There are lies, damned lies, and statistics. If we want to estimate the risk for crane crashing for all kinds of causes, we have a situation different from calculating the risk for tipping. I believe the insurance companies must have mathematical models helping them to settle insurance prices for crane crashes, where the crash may be due to metal faults in the crane construction, unusual storms, human bypass of the regulations ensuring not tipping, and many other reasons. However, these models are not to be trusted like the tipping model. How can we know that all kinds of causes have been taken into account in the models? How are probability estimates for different thinkable events derived? Judging crash risks for all kinds of causes the construction workers and the contractors shall be aware of their opposite interests and not believe too firmly in conclusions from mathematical modelling with a lot of build in flexibility. This, together with theorem 1, leads us to conclusion 1.

Conclusion 1: Since, 1) mathematical models are gaining increasing importance, and 2) neither blind scepticism nor blind confidence suffice, insight in the diversity of mathematical models is needed

This of course raises the question about the sort of insight needed and how to establish it. However, before coming to that, we turn to a main obstacle for getting the insight in the diversity of mathematical models.

Theorem 3: Mathematical models look alike

For the sake of convincing, you look at the following, not models, but equations:

$$(a+b)(a-b)=a^2-b^2$$

$$c^2=a^2+b^2$$

$$E=mc^2$$

$$F=-kx$$

$$R=PC$$

For ordinary people these five equations look very much alike. They are all letter expressions equating the two sides of the equality sign. However, the similarity of the five equations is superficial. In the first equation a and b represents arbitrary numbers. The equation is an algebraic identity, a consequence of how we operate algebraically, and as such generally valid. The second equation is the Pythagorean Theorem valid for right-angled triangles in a plane. But not for, e.g., right-angled triangles on a sphere. The third equation, due to Einstein, demonstrates the equivalence of mass and energy and is believed to be universally true. The forth equation is Hook´s law expressing how a spring elongates due to a force on it. This is a first order approximation for small elongations and not at all a universal law. Finally the fifth equation belongs to risk analysis. It expresses the risk of an event as the probability of the event times its consequence. The equation is disputed and is to be understood as a definition of risk.

The five equations deal with very different realities although they look very much alike for ordinary people. The same is true for mathematical models. The ways they relate to different realities are very different, although it is overlooked, not only by ordinary people, but also by many professionals. Which lead us to theorem 4.

Theorem 4: In the school, in the disciplines and in the society there exists a cover up of the differences in characters and functions of different mathematical models

As a school related example I will mention how Ohm´s law and Coulomb´s law were presented in my high school lessons in physics. Ohm´s law tells that the voltage difference, V, between the ends of a conductor and the current in it, I, is proportional. That is: V=RI, where the proportionality constant, R, is the resistance of the conductor. Coulomb´s law tells that the force, F, between two charges, q_1 and q_2, are equal to a proportionality constant, k, times q_1q_2, divided by r^2, r being the distance between the two charges. The two laws are sketched in figure 2.

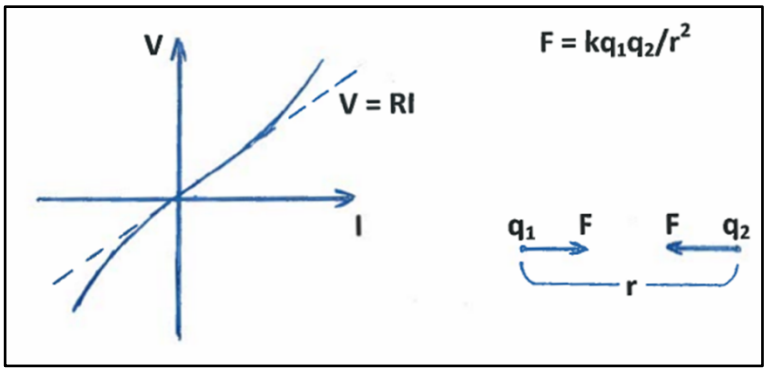


Figure 2: Ohm´s Law and Coulomb´s law. V is voltage difference, R resistance, I current, F force, k proportionality constant, q_1 and q_2 charges, and r is distance between q_1 and q_2.

My point here is that the two laws were presented as having the same quality. They were both important equations in electricity theory useful in problem solving. However, in figure 2 it is illustrated that Ohm´s law is nothing but a first order approximation. Since I is zero when V is zero and I change sign when V changes sign the function V(I) must go through (0,0) and be symmetric around (0,0). And such a function can in most physical cases be approximated by a straight line not too far from (0,0). Ohm´s law is a descriptive ad hoc law like the above-mentioned Hook´s law. We should not be surprised experiencing deviations at large values of V and I. Counter wise, we should be very surprised finding proven deviations from Coulomb´s law. Coulomb´s law is interwoven with Maxwell`s basic equations for electrodynamics as a whole. The power 2 of r is, e.g., related to space having 3 dimensions. Thus, substituting r^2 with $r^{1.99999}$ would imply that the whole of electrodynamic theory should be fundamentally revised. However, as mentioned the different epistemological quality of the two laws were not pointed out in the school.

As for a discipline related example, consider figure 3.

Figure 3: Astronomy and Urban Geography. M_1 is mass of globe 1, M_2 mass of globe 2, m mass of meteor, F_1 force on m from M_1, F_2 force on m from M_2, r_1 distance between M_1 and m, r_2 distance between M_2 and m, S_1 size of town 1, S_2 size of town 2, p a person, r_1 distance from S_1 to p, and r_2 distance from S_2 to p.

A part of astronomy is studying the motion of globes due to their gravitational interactions. In the figure, a very simple case is illustrated: A meteor is at rest between two globes. We can then ask: To which side does the meteor with mass

m fall? From Newton´s gravitational law we know the gravitational attraction force, F_1, on m from globe 1, width mass M_1 and distance r_1 from m to M_1, to be $F_1=GmM_1/r_1^2$, where G is the universal gravitational constant. Similarly the gravitational force, F_2, on m from globe 2 is given by $F_2=GmM_2/r_2^2$. Thus, the answer to our problem is: If $M_1/r_1^2>M_2/r_2^2$, implying $F_1>F_2$, the meteor falls towards globe 1. If $M_1/r_1^2<M_2/r_2^2$, implying $F_1<F_2$, the meteor falls towards globe 2.

A part of urban geography is studying persons buying behaviour. In the figure, a very simple case is illustrated: A person p lives in between town 1 with size S_1 at distance r_1 and town 2 with size S_2 at distances r_2. We can then ask: In which town does the person p buy goods? In urban geography, the first guess, according to the so called gravitational law of urban geography, would be: If $S_1/r_1^2>S_2/r_2^2$ the person buys goods in town 1. If $S_1/r_1^2<S_2/r_2^2$ the person buys goods in town 2. However, the reference to Newton´s law of gravitation is covering up the differences between mechanical astronomy and urban geography. In urban geography it is debated what should be used as a measure of the size of a town. Is, e.g., the number of inhabitants, the number of shopping square meters or the number of parking places the relevant measure for S in the law? In Newton´s gravitational law, M is without discussion a mass. In urban geography, the power of r is empirically adjusted from 2 to, e.g., 1.9 or 1.7 for different localities. In Newton´s gravitational law, as in Coulomb´s law, the power 2 is, via the rest of Newtonian mechanics, related to space having 3 dimensions, and thus not adjustable. Concentrating empirical data in the form of a formula as done in urban geography may be a smart economical idea with many advantages. But it does not resemble Newton´s gravitational law. The two examples lead us to conclusion 2.

Conclusion 2: A DISTINCTION between THEORY DERIVED mathematical models and AD HOC mathematical models should be presented in the educational system

The increasing scientific management of technology and society, supported by increasingly powerful information technologies, has led, and leads to increasingly widespread use of mathematical models. Since neither blind scepticism nor blind confidence suffice, the general public needs a cognitive compass helping judging the diversity of the mathematical models. Moreover, since there are limits for the engagement of the single individual, a foundation build on well-chosen examples should be established in the educational system. Furthermore, the apparent similarity of the mathematical models for most people, instead of covering up their differences in characters and functions, in the school, in the disciplines and in the society, as a beginning, a crude

distinction, between *theory-derived mathematical models* like Coulomb´s law and the astronomical model on the one side, and *ad hoc mathematical models* like Ohm´s law and the urban geography model on the other side, should be presented in the educational system.

With theory-derived mathematical models is meant models interwoven width or derived, as a special application, from a broader mathematically formulated theory. They are deductive in nature. In contrast, ad hoc mathematical models inductively summarizes an existing empirical reality in a compact mathematical languish without reference to more global theories. The empirical justification of an ad hoc model comes from the apparent context, whereas the theory-derived model gets its empirical justification from the broader empiric scene of earlier model applications of the theory, having consolidated the theory.

Is the distinction unnecessary crude? Do most mathematical models perhaps lie between the two extremes? Yes, but in order to orientate yourself you have to know the extremes. Do I (as physicist) deprive everything besides physics theoretical status? No, but it is correct that the development of physics for good and bad in particular has been interwoven with that of mathematics. Other theories are not in their outset formulated in mathematics. Apart from geometry, combinatorics, and population genetics epistemological character of mathematical models depends on their reference or not to mathematical formulated theory.

However, the distinction cannot be used to evaluate, whether specific models are hiding or guiding. When modelling you always look apart from many aspects making modelling both possible and simple. Thus, theory-derived models may be too idealized to be trusted. Moreover, ad hoc models may be trustworthy due to richness of input data approaching a detailed map of the phenomenon to be modelled. So, what is the usefulness of the distinction?

Theorem 5: Theory-derived mathematical models can be subject to BOTH EMPIRICAL AND THEORETICAL CONTROL, ad hoc mathematical models can ONLY be CONTROLLED EMPIRICALLY

The value of the distinction is that it makes it clear, that some mathematical models, the theory-derived models, besides of the possibility of evaluating them by comparing with empirical data, also can be evaluated by theoretical considerations. Evaluating ad hoc models is, in contrast, restricted to be done by empirical control only.

Figure 4 gives an illustration of what is meant by theoretical control.

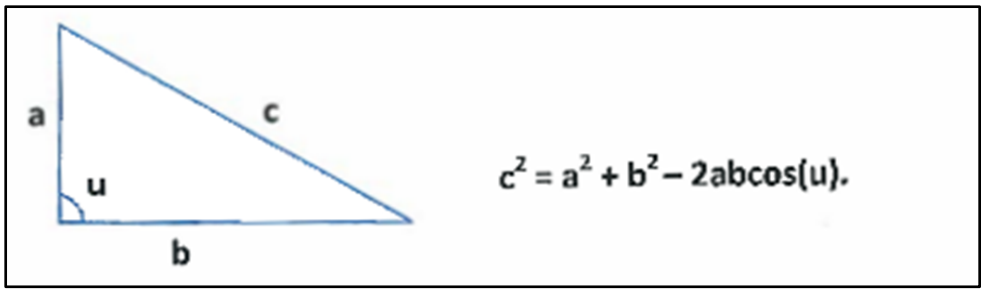


Figure 4: Theoretical control of consequences of deviation from a right angel. a, b and c are sides in an almost right angled triangle. u is the almost right angle.

Let us imagine that the almost right-angled triangle of figure 4 is part of a construction model where it is considered right angled, and thus $c^2=a^2+b^2$. Hereby we introduce a failure in our construction calculations since no triangles are exactly right angled in practice. If we only had access to the mathematical model $c^2=a^2+b^2$ we were restricted to make loose estimates of the consequences of u varying between, e.g., 89^0 and 91^0. But, since the mathematical model "right angled triangle" is a special case of the broader theory of all sorts of triangles, the consequences of the variations of u for the construction calculations can be estimated precisely using the more general formula for triangles: $c^2=a^2+b^2-2abcosu$. Perhaps with the result that the idealisation $c^2=a^2+b^2$ was justified considering the purpose of the model.

We had another example of possibility of theoretical control in the crane example. How small does the weight of the crane construction need to be, compared to the load and the contra weight, in order to justify not considering it when judging the risk of crane tipping? Although more elaborately, this can be calculated using the same torque rules from Newtonian mechanics as used deriving the idealised model result, $CD_1>LD_2$ (see figure 1). However, if we ask about the risk for a crash of the crane for any cause, no theoretical framework exists to help us judge how we can trust the necessary simplifications made, making a mathematical model estimating the risk. Such a model has to be controlled by confronting it with accident statistics. That is by empirical control. The tipping problem can be placed inside the framework of a mathematical formulated theory in a well-defined manner. Therefore, we are aware of what we ignore in order to make our model simple. Moreover, the induced failures due to the simplifications can be judged currently by the theory. In the crash problem, it is not a question of judging conscious looking apart, the question is rather what we have over looked. Only empirical data can help us answer that; often with the answer that we have over looked something, without knowing what.

As mentioned, the distinction between theory-derived mathematical models and ad hoc mathematical models cannot be used to distinguish between trustworthy and non- trustworthy models. The distinction only implies different conditions

for evaluating how trustworthy the models are. Considering this and the abstractness of the distinction one could argue that it is a tame message not worthwhile communicating in the educational system for democratic reasons? But, a closer look makes it perhaps not look that tame.

Theorem 6: Theory-derived mathematical models can be criticised in public by independent experts. In order to criticise ad hoc mathematical models accession to data is needed

The trustworthiness of theory-derived mathematical models versus ad hoc mathematical models cannot be judged just epistemologically. However, the different terms for controlling the two kinds of models, nevertheless often gives sociological reasons for a higher believe in the theory-derived models than in the ad hoc models.

Empirical control of ad hoc mathematical models implies access to data. And this access can be monopolised by owners of the data. Mathematical models in combination with IT most often operate with huge amounts of data. Thus, the collection of data can be expensive, making it hard to carry out empirical control by outsiders.

Theoretical control of theory-derived mathematical models demands insight in theory. This insight is most often reserved for experts with special educations. But among them counter-experts may exist. Moreover, their willingness to deliver theoretical control of a model depends perhaps more on commitment than on money.

Thus, after all, in general, theory-derived mathematical models are more open to public critic and discussion than ad hoc mathematical models. The higher possibility of critic and control from outside experts makes it more risky to tinker with theory-derived mathematical models than with ad hoc mathematical models. Thus, the construction worker have good reasons to believe more firmly on the calculation of the condition for a crane not tipping than on the calculations of crash risks at all.

Besides the final conclusion 3, I hereby have ended the attempt of presenting my points on the hiding or guiding problem of mathematical models using pedagogically chosen examples. As mentioned in the introduction I will here in the end of the paper, before presenting conclusion 3, use the developed concepts to compare my validation of the three, less pedagogical and more societal important mathematical models, mentioned in the introduction.

The GPS-system, Climate models, and Macroeconomic forecasting

I am not an expert in neither the Global Positioning System, or climate models, or macroeconomic forecasting. I have my information about these models from, e.g., Wikipedia. Nevertheless, due to many years of experiences with different kinds of mathematical models and the concepts presented above, I dare judge large qualitative differences between the three models.

The GPS-system operates by triangulating between a receiver and satellites orbiting around 20000 km above the surface of the earth. In order to make the triangulations you must calculate the motion of the satellites using Newtonian mechanics. But you also have to use both special and general theory of relativity! We need to know the positions of the satellites when they send their radio signal to us. However, judged from our position the clock in a satellite is not running with the same speed as our clock. The special theory of relativity tells us that the satellite clock runs slower than ours does because the satellite moves relative to us. The general theory of relativity tells us that the satellite clock runs faster than ours does because the gravitational field from the earth is weaker at the satellite than on the surface of the earth. The combined effect is that we experience a clock in the satellite identical to the clock in our hand running faster. Therefore, the clocks in the satellites are technically adjusted to run slower in order to show our time. Without that adjustment, the GPS-system would not function.

The GPS-system is put under empirical control of all of us using it. It works with astonishing accuracy. But how was it developed to work that precisely. By trial and error? No, without being informed about details, I am convinced that the developing process has been steered by a lot of theoretical control of the mathematical models involved. It is hard to imagine the development of the GPS-system without the theory-derived mathematical models involved, and their current adjustments guided by theoretical control.

Climate models are not throughout theory-derived like the mathematical models of the GPS-system. Parts of them are theory-derived. But they also include more descriptive ad hoc parts. The greenhouse effect due to CO_2 in the atmosphere is theoretically understood for more than hundred years. The selective absorption of electromagnetic radiation of different wavelengths of the CO_2 molecule could be derived theoretically. And the essential thermodynamically consequences of that for the temperature on the earth were foreseen theoretically. However, the temperature variation on the earth caused by fossil fuel burning during the last hundred years is not equally easy to derive theoretically. Besides being influenced by the greenhouse effect the temperature of the atmosphere is, e.g., also influenced by the stirring in the oceans. The heat capacity of the atmosphere equals only the heat capacity of 10 meter of the 4000 km deep oceans. Therefore, there is room for much heat exchange between the atmosphere and the oceans

depending on the not very well understood stirring in the oceans. This heat exchange has to be taken into account more ad hoc like, making forecasting of the temperature on earth rather uncertain.

The theory-derived understanding of the greenhouse mechanisms gives me reason to believe in the order of magnitude of the forecasted temperature increase from climate models. But, the heavy debate about the issue, contrary to a not existing public debate about the models behind the GPS-system, does not surprise me. The debate is of course due to the climate problem being a severe problem with a lot of interests involved in it. But, it is also because the climate models are a mixture of theory-derived models and ad hoc models.

The macroeconomic forecasting models are also a mixture of theory-derived models and ad hoc models. There are many model assumptions in them like using Ohm´s law or as in urban geography. And there are many problems connected to the empirical estimation of parameters in these ad hoc assumptions. In these aspects they resemble the climate models. However, they differ from the climate models and the models of the GPS-system, their theory-derived parts being based on a conceptually different kind of theory. In order to compare the macroeconomic forecasting models with the GPS-models and the climate models, we need to distinguish between different conceptual uses of the word "theory", besides distinguishing between theory-derived mathematical models and ad hoc mathematical models.

Theory as believed reality and theory as assumption

Most physicists have a sort of platonic view of the world. For us, a theory is even more real than facts. Empirically registered phenomenological facts may overshadow the essential laws underneath the phenomenological facts. Thus, we believe Newtonian mechanics, special theory of relativity and general theory of relativity to be real. The GPS-system is functioning well because these theories are real.

This does not mean that we believe that the theories are ontologically true, that is, universal and ever-lasting. How *ontological truth* may be established has been discussed by different philosophers with different views, e.g., the logical positivists, Karl Popper and Thomas Kuhn. However, what physicists primarily are interested in when making theory-derived mathematical models is the *pragmatic truth* of theories, that is, their trustworthiness inside specified limits. Have the theories of relativity falsified Newtonian mechanics? Ontologically speaking, yes, since it is possible to formulate problems inside Newtonian mechanics that cannot be formulated inside the theories of relativity, and vice versa. But, pragmatically speaking, no. On the contrary, the theories of relativity

have strengthened the truth of Newtonian mechanics by specifying its operational domain. Thus, physics students at universities are taught to shift between and combine the paradigms of Newtonian mechanics and the special theory of relativity depending on the problems to be solved. Unfortunately, Kuhn´s characterisation of Newtonian mechanics and the special theory of relativity, as not comparable ontological paradigms, has fostered the misunderstanding outside physics that physicists may believe in different pragmatic realities. Thus, the public and many economists are misguided, by the interests of the philosophers in ontological matters, to underestimate how huge the epistemological differences between the macroeconomic forecasting models and the theory-derived models in the GPS-system are.

Like the climate models, the macroeconomic forecasting models are composed of a mixture of theory-derived mathematical models and ad hoc mathematical models, the ad hoc models often being weak because of too few empirical data to control and adjust the models. Besides, in economy the word theory is used differently from the use of it in connection with climate models and the GPS-system. Economists use the word the way it is used by pure mathematicians, not by physicists. By a theory is meant an assumption and the logical deductions that can be made from it. Plane geometry is an example of a mathematical theory. Assuming the earth flat, the theory gives results in exact concordance with the theory when developing maps. But of course the theory cannot be used to navigate over large distances on the earth surface, since the assumption made when choosing the theory is wrong. Likewise, macroeconomic forecasting mathematical models are deduced from not verified theoretical assumptions. They may be deduced from, e. g., more neoliberal assumptions or more Keynesian assumptions. In physics, a free choice of theory does not exist. The macroeconomic forecasting models are more prescribing than describing. They prescribe economical actions realizing political goals hidden in their theories. The value of theoretical control in economic models is that it controls their internal consistency. It does not control their trustworthiness. If theoretical control shall ensure trustworthy descriptions of reality, given by a theory-derived mathematical model, the theory have to be believed as a reality due to many and diverse empirical verifications of deductions from the theory. And the empirical verifications of the macroeconomic forecasting mathematical models are few and weak. Competing macroeconomic forecasting models may be a useful qualitative way of ordering and clarifying discussions of the many mutual interactions of factors in the economy. But I do not believe in them as quantitative descriptions of the economic reality.

As mentioned, I am not an expert in neither the GPS-system, or climate models, or macroeconomic forecasting. Besides being a physicist, I have had amateur

experiences with a variety of mathematical models and reflections that are more general on mathematical models in society. This is my background for daring the above judgements comparing the GPS-system, climate models and macroeconomic forecasting. Likewise, it is not necessary for the general public to know details in specific mathematical models in order to have some judgement about their validity and the way the validity is controlled. It is more important being informed about a variety of different simple cases to compare with. This leads me to the final conclusion:

Conclusion 3: 1) It is democratically needed that the educational system supports the public judgement of the diverse character of mathematical models. 2) General philosophical considerations and/or analysis of interests are not sufficient. Comparative, interdisciplinary experiences with mathematical models have to be offered

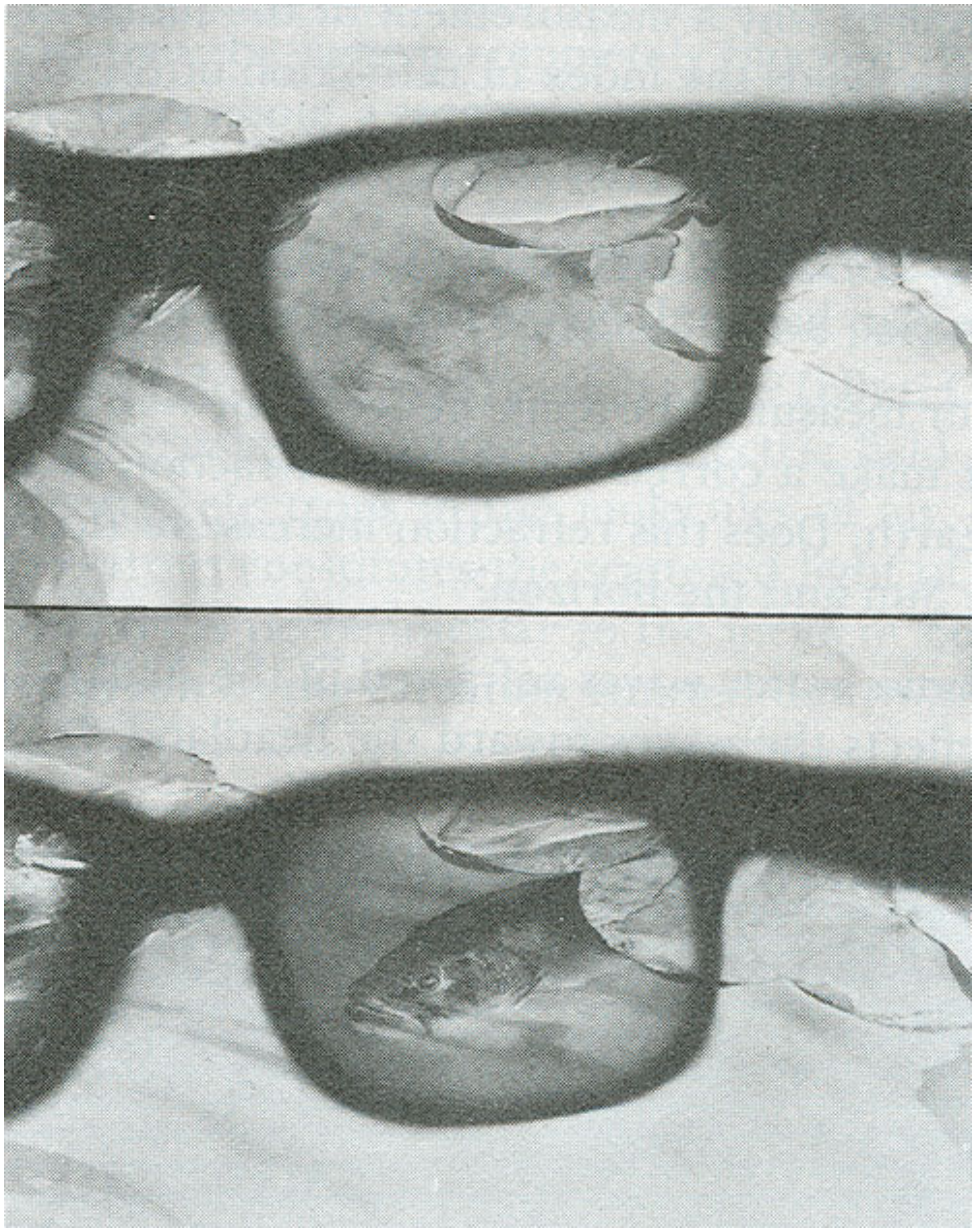


Fig. 37.32 When viewed through ordinary sunglasses (top), the fish is almost completely hidden by the glare reflected by the water. When viewed through Polaroid sunglasses (bottom), the fish becomes visible because the glare is reduced.

“Modsætningen til at overse er at se bort fra”

Den følgende kommentar, skrevet på opfordring, har været bragt i MONA, 2019-1.

Naturvidenskab og dannelse

Jens Højgaard Jensen,
IMFUFA, INM, RUC

Kommentar til Jens Dolin: "Enkeltfag eller fagintegration i naturfagene?", MONA, 2018-4.

Med erfaring fra at have været prorektor på RUC og studieleder for og projektvejleder ved den naturvidenskabelige basisuddannelse på RUC gennem en årrække genkender jeg meget i Jens Dolins (JD) artikel. Fx hvordan tværfaglighed vanskeliggøres af at enkeltfagene lukker sig om sig selv, og hvordan realiseringen af ideale uddannelsesambitioner vanskeliggøres af rent administrative hensyn og interessevaretagelseskampe. Men jeg kan ikke genkende JD's måde at koble naturvidenskabelig dannelse og tværfaglighed på. Der er mange grunde til at arbejde for tværfaglighed i uddannelserne. Men at "tværfaglighed, hvor det vigtigste i undervisningen ikke er fagene, men sagen", skulle være særlig egnet til at danne elever og studerende til et myndigt og meningsfuldt liv, og at enkeltfagene derimod i sig er instrumentelt orienterede, genkender jeg ikke. Kommentaren her handler derfor om hvordan jeg synes modsætningsparret enkeltfag/fagintegration og modsætningsparret instrumentelle hensyn/dannelseshensyn forholder sig til hinanden.

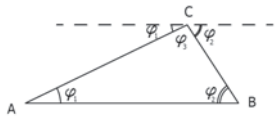
Når der tales om dannelse, synes jeg, på linje med JD, at det vigtigste at indholdsudvikle hænger sammen med det centraleuropæiske/nordiske dannelsesbegreb som drejer sig om det personlighedsformende i en persons uddannelsesforløb, til forskel fra det snævrere angelsaksiske begreb "scientific literacy" rettet mod mere specifik viden og specifikke færdigheder. Med fokusering på det personlighedsformende er det mere naturvidenskabernes mulige dannelsesbidrag end naturvidenskabelig dannelse det så drejer sig om. Samfundsvidenskaberne kan bidrage med manøvredygtighed i forhold til interessemodsætninger som del af dannelsen. Humaniora og kunst kan bidrage med udviklingen af moralske værdier og eksistentielle sandheder som dele af dannelsen. Tværfagligt projektarbejde rettet imod en sag kan bidrage med udviklingen af gåpåmod og struktureringsevner i komplekse sammenhænge og politisk dannelse hvis sagen er samfundsrettet. Hvad kan naturvidenskaberne bidrage med? Piet Hein har karakteriseret medlemmerne af C.P. Snows to akademiske kulturer, den humanistiske og den naturvidenskabelige, som henholdsvis "kultister" og "teknoter".

Som forfatter og tidligere fysikstuderende havde han selv et ben i hver lejr som han ligesom C.P. Snow med beklagelse fandt adskilt af en kulturkløft. Hvis nogle fandt at ordet "kultist" mindede om "okkult", og ordet "teknot" om "idiot", havde de forstået sagen rigtigt. Så måske burde naturvidenskaberne bidrage med at supplere elevers og studerendes erfaringsunivers angående eksistentielle sandheder med et udbygget erfaringsunivers angående objektive sandheder – med eftertryk på erfaringer frem for filosofiske abstraktioner. Ikke mindst i en tid hvor "oplysningen" er under pres, er det måske vigtigt. Hvordan kan matematik, fysik, kemi, biologi og naturgeografi – integreret eller som enkeltfag – bidrage?

Når jeg bruger udtrykket objektive sandheder, skal det ikke forstås ontologisk, altså som universelle og eviggyldige sandheder. Som mange andre naturvidenskabsudøvere er jeg skeptisk over for tendentielt at diskutere videnskab som religion. Både i forhold til videnskab og religion er jeg agnostiker. Med objektive sandheder tænker jeg på sandheder der ikke er socialt konstruerede, praktisk talt. Selv er jeg fysiker og fysikdidaktiker. Som fysikdidaktiker er jeg generet af den angelsaksiske tradition for at reducere naturvidenskabsdidaktik og pædagogik til en naturvidenskabsagtig måle/veje-sag. Jeg synes først og fremmest at naturvidenskabsdidaktik er et socialt konstrueret refleksionsrum til vedligeholdelse af didaktisk og pædagogisk eftertænksomhed. Omvendt generer det mig som fysiker når folk uden for fysik gennemgående misforstår Kuhns kritik af Popper og de logiske positivister. Kuhn kritiserer deres ontologiske ambitioner. Man kan ikke hævde at paradigmet Einsteins relativitetsteori er nærmere en ontologisk sandhed end paradigmet Newtons mekanik da der ikke findes en ontologi til at afgøre sagen. Men Kuhn er selvfølgelig som uddannet fysiker klar over at Einsteins relativitetsteori pragmatisk set har større praktisk rækkevidde end Newtons mekanik. Og pragmatisk set er der ikke tale om konkurrerende paradigmer. Fx fungerer GPS-systemet blandt andet i kraft af et velfungerende samvirke mellem geometri, Newtons mekanik, Einsteins specielle relativitetsteori og Einsteins generelle relativitetsteori. Pragmatisk set leverer både geometri, Newtons mekanik og Einsteins specielle og generelle relativitetsteorier objektive sandheder til opbygningen af GPS-systemet. Kort sagt: De sandheder jeg oplever som fysiker og som fysikdidaktiker, er af meget forskellig karakter. Og for ikke at ende som enten teknot eller kultist er det måske i bredere almindelighed væsentligt at have erfaringer med begge slags sandheder.

Er det så tværfag eller enkeltfag der skal til for at elever og studerende erfarer selvoplevede objektive sandheder i naturvidenskabsundervisningen? Hvis der med naturvidenskab menes fagrækken matematik, fysik, kemi, biologi og naturgeografi, er svaret komplekst.

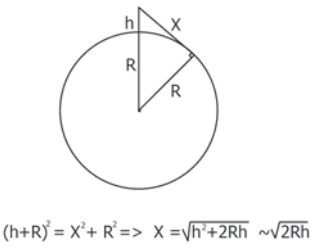
På figur 1 er beviset for at vinkelsummen i en plan trekant er 180°. Den stiplede linje tænkes at være parallel med linjestykket AB. Summen af vinklerne ved A, B og C, $\varphi_1 + \varphi_2 + \varphi_3$, genfindes da som 180° ved C.



Figur 1. *Bevis for at summen af vinklerne i en vilkårlig plan trekant, $\varphi_1 + \varphi_2 + \varphi_3$, er 180°.*

Jeg trækker beviset frem som illustration af kernen i matematik i et par tusinde år siden de gamle grækere: Man kan rationelt tænke sig til objektive sandheder. Beviset handler ikke om den konkret tegnede trekant, men om enhver trekant da det alene baserer sig på at der er tale om en trekant – sådan som beviser fungerer i matematik, modsat fx beviser af hvem der var forbryderen, eller hvilken bakterie der forårsagede sygdommen. Matematiske beviser bliver stående når de først er leveret, modsat sociale konstruktioner. Og oplevelser heraf er selvsagt personlighedsformende. Det personlighedsformende kan således være tæt knyttet til et enkeltfag.

På figur 2 optræder matematik modsat på udebane som modelleringsværktøj. Der er tale om det geometriske/fysiske problem at skaffe sig en formel til at vurdere hvor langt væk horisonten er når der skues ud over havet. Indsættes h = 2 m og R ~ 6000 km i X = (2Rh)½, fås X ~ 5 km. Det er altså afstanden til horisonten når man står på strandbredden. Hvis vi står på toppen af Rundetårn med øjet ca. 16 gange så højt over havet som på strandbredden, ses det da af formlen at X er (16)½ = 4 gange større. Horisonten set fra Rundetårn er altså ca. 20 km væk.



Figur 2. *Udregning af hvor langt væk horisonten er. R er Jordens radius, h er øjenhøjden over havoverfladen, og X afstanden til horisonten.*

Jeg trækker dette matematiske modelleringsproblem frem som eksempel på hvordan matematik typisk fungerer i sammenhæng med fysik. Med Einsteins ord: Make it as simple as possible, but not simpler. Vi har idealiseret Jorden til at være kuglerund, undladt at skelne imellem X og afstanden til horisonten langs jordoverfladen samt sat h^2 til nul sammenlignet med 2Rh. Men vi har styr på tilnærmelserne så nøjagtigheden af horisontformlen kan vurderes. I den forstand er horisontformlen en objektiv sandhed og ikke en social konstruktion. Oplevelser af, ved hjælp af matematisk mo-

dellering, at kunne forudsige empirisk målbare forhold har selvsagt også for nogen haft personlighedsformende betydning – ikke siden de gamle grækere, som det er tilfældet med matematiske beviser, men siden den videnskabelige revolution. Her ligger næringssubstratet imidlertid ikke i et enkelt fag i gymnasiet eller folkeskolen. Det ligger mest oplagt i et tværfagligt samarbejde imellem matematik og fysik om matematisk modellering. Et samarbejde som vanskeliggøres af fagopdelingen (og i øvrigt også af adskillelsen af matematikdidaktik og naturfagsdidaktik fra hinanden).

Matematiske beviser og matematisk modellering kan bidrage med oplevelser af hver sin udgave af objektiv sandhed. Endnu mere betydningsfuld er imidlertid oplevelser der henter deres sandhedsværdi mere direkte i empiriske erfaringer end i rationelle overvejelser. Sådanne oplevelser har selvfølgelig eksisteret for alle til alle tider. Undervisningsproblemet er at formidle oplevelser af forskellen imellem generaliseringer ud fra enkeltstående erfaringer og tilvejebringelse af større eller mindre evidens afhængigt af anvendte metoder til erfaringsindsamling. Jeg nævnte i en tidligere artikel i MONA (2017-2) at jeg på et tidspunkt har været projektvejleder for en gruppe på den naturvidenskabelige basisuddannelse på RUC som interesserede sig for jordstråler, hvor jeg fik gruppen overtalt til at dreje projektet imod en komparativ modstilling af jordstråler og radioaktive stråler. Man kan hverken tage, se eller føle på hverken jordstråler eller radioaktive stråler. Hvordan kan vi fysikere da vide at der er evidens for det ene og ikke for det andet? Er der ikke blot tale om en fysikerfordom? Eksemplet viste mig at det der rykkede på de studerendes forståelse af hvad der er evidens for, og hvad ikke, ikke så meget var argumenter af bestemte slags som det var deres integrerede erfaringer gennem hele projektforløbet – på samme måde som man bredt i samfundet tydeligt kan opleve at stillingtagen til astrologi, alternative behandlinger m.m. mere afhænger af folks større eller mindre naturvidenskabelige forankringer end af modtaget videnskabsteori- undervisning. Kort sagt: Empirisk evidens er ikke så klar og entydig en størrelse. Den skal erfares i dens mange versioner. Det inviterer til et tværfagligt samarbejde herom mellem de empiriske/eksperimentelle fag fysik, kemi, biologi og naturgeografi i stedet for at hvert fag arbejder med problemstillingen uafhængigt af hinanden. Men det forudsætter at bredden af faglige input er til stede og ikke svækkes af administrative eller andre grunde, fx faglige interessevaretagelseskampe. Ifølge Smithers og Robinson (Smithers & Robinson, 2006) medvirkede oprettelsen af et sådant fag i folkeskolen i England imod hensigten til en faldende søgning til STEM-uddannelserne (Science, Technology, Engineering, Mathematics) og øget søgning til sundhedsuddannelserne fordi realiseringen gennemførtes af lærere med helt overvejende biologisk baggrund.

De forskellige naturvidenskabelige fag har – hver for sig og fagintegreret – forskelligt at bidrage med til udviklingen af elever og studerendes myndighed i kraft af dømmekraft om hvad der i praksis kan regnes for objektive sandheder, og hvad ikke. De

naturvidenskabelige fag er ikke ens konstitueret i forhold til tværfagligt samarbejde. I en nylig artikel (Jensen & Jankvist, 2018a) har Uffe Thomas Jankvist og jeg forsøgt at karakterisere fags forskellige konstituering i det hele taget. Samtidig har vi i en anden artikel (Jensen & Jankvist, 2018b) forsøgt at give et bidrag til fremme af tværfagligt samarbejde fagene imellem på trods af deres forskelligartethed.

Referencer

Jensen, J.H. & Jankvist, U.T. (2018a). Disciplines and ways of perception: linking interdisciplinarity and competences. I: T. Sibbald (red.), *Teaching interdisciplinary mathematics* (s. 119-132). Champaign: Common Ground Publishing.

Jensen, J.H. & Jankvist, U.T. (2018b). Disciplinary competence descriptions for external use. *Nordisk Matematikdidaktik (NOMAD)*, 23(2), 3-24.

Smithers, A. & Robinson, P. (2006). *Physics in schools and universities II: patterns and policies.* Carmichael Press, University of Buckingham.

Den følgende artikel har Martin Niss og jeg uden held forsøgt at få optaget i flere forskellige engelsksprogede tidsskrifter. Vi har haft svært ved at finde et tidsskrift, som ville acceptere mixet mellem en filosofisk-analytisk diskussion og gennemgangen af fysikopgaver som illustration. Vi synes selv, at det er en god og væsentlig artikel.

Nomological versus Causal Reasoning When Learning Physics

1. Abstract

This paper considers a distinction between two types of reasoning used in physics teaching and learning. *Nomological* reasoning is when you point out how a physical phenomenon is a manifestation of a general pattern under the circumstances present. *Causal* reasoning is when you point out causes for the phenomenon in the circumstances present. Teacher awareness of this theoretical distinction is useful since students normally have greater difficulties with understanding and applying nomological than causal reasoning. So, it is important to be able to communicate the kind of reasoning we want students to perform as well as enter a dialogue with them about different types of reasoning. The aim of the paper is to provide illustrative examples that can help communicate the distinction by showing causal and nomological problem solving approaches to four different physics problems. The problems are chosen so that they illustrate four different relations between the two types of reasoning.

2. Keywords

Scientific reasoning, scientific explanation, physics education, problem solving

3. Introduction

This paper considers a distinction between two types of reasoning used in physics teaching and learning.[1] When reasoning *nomologically* (the word "nomos" comes from Greek and means rule or law) you explain a physical phenomenon (e.g., an event, a fact or a state of affairs) by referring to a general pattern and show that the phenomenon is a manifestation of the pattern in the present situation. Hence, the occurrence of the phenomenon can be inferred from the general pattern combined with specific characteristics of the phenomenon. When reasoning *causally* you show how the phenomenon in question is the effect of the causes present in the situation. So, the phenomenon is placed in a causal chain and the mechanism that brings the phenomenon about is specified. General laws may be involved in causal reasoning because the operation of the mechanism is often in accordance with a law. In short, nomological reasoning is when you point out how the phenomenon is a manifestation of a general pattern under the circumstances present. Causal reasoning is when you point out causes for the phenomenon in the circumstances present.

This distinction is not new. It can be found in quite a few physics textbooks and several philosophers have been debating the status of causal reasoning in science for a long time as well as whether physics are using and should be using something like nomological reasoning (see Besson 2010). The most prominent philosophical representatives for the two approaches are Carl G. Hempel (1965) and Wesley Salmon (1984) respectively. Science educators point out that pupils and students tend to use causal reasoning, while physicists and physics teachers employ and request nomological reasoning. Besson sums up the educational research on the differences between reasoning done by students, on the one side, and those employed by scientists and teacher, on the other, in the following way:

[1] Kirsten Ringgaard Jensen has treated this distinction and coined the names in her Master thesis of 2004 (Ringgaard Jensen, 2004).

"Students mainly

- reason in terms of changes;
- are interested in transitory phases, and in the processes leading to a given situation;
- consider causes and effects in temporal succession;
- need intuitive models and physical analogies.

While often instead teachers:

- reason in terms of equilibrium or steady-state situations;
- are interested to the maintenance conditions for a situation;
- consider relationships between quantities at a given time;
- propose formal laws and abstract mathematical models." (Besson 2010, p. 243)

As way of an example, Psillos (1995) points out that when physics teachers treat direct current circuits, their reasoning "is based on the study of functional relationships between their basic macroscopic parameters. The reasoning which is expected presupposes a systemic view and is a-causal." (Psillos 1995, p. 67) In contrast, student reasoning is often causal, and they look for causes that would take a circuit to a new state. In general, many students are not satisfied by the mere application of formal laws and functional relations in systemic or holistic descriptions and they request causal mechanisms.

We suggest that most physics teachers would agree that nomological explanations are the essence of physics because they show how general laws of nature imprint themselves on the physical world. Hence, the distinction is relevant for physics educators because it can help articulate for both teachers and students this characteristic of physics. Moreover, since the distinction characterizes two different reasoning patterns, it can be used to analyze students' actual reasoning patterns and how these patterns sometimes differ from those of the teacher.

Since nomological reasoning is central to reasoning in physics, but at the same time causes difficulty for students, we are facing an educational challenge. It is important to be able to communicate the kind of reasoning we want students to be able to perform as well as enter a dialogue with them about different types of reasoning. The first type of reasoning is fairly easy to grasp and describe. This is the type of reasoning, that the philosopher Wesley Salmon captures, when he says that "to give a scientific explanation is to show how events fit into the causal structure of the world" (Salmon 1984, p. 19). Due to the abstract and somewhat unwieldy nature of the nomological reasoning, it is more difficult to describe and communicate. What does it mean to use formal laws and functional relationships and give a holistic description? Therefore, the aim of the paper is to provide illustrative examples that can help communicate the distinction in analogy with the saying "I cannot define my wife, but I can recognize her when I see her." We show causal and nomological problem solving approaches to four different physics problems. We have chosen the problems in such a way that they illustrate four different relations between the two types of reasoning.[2]

The words "nomological" and "causal" stem from the philosophical literature on the nature of scientific explanation, (Hempel 1965, Nola & Irzik 2005; Salmon 1984; Salmon 1990) where examples from physics

[2] These problems are from written exams for the courses "Problem Solving in Physics" at Roskilde University and can be found in Højgaard Jensen (2018). Problem 1 can be found on page 155, Problem 2 on page 215, problem 3 on page 205 and problem 4 on page 160.

are used to inform philosophical discussions on such questions as whether physical explanations are causal or nomological in nature. Our goal is different, namely to use our theoretical distinction to inform an analysis of students' ways of reasoning. We use "nomological" and "causal" as descriptive labels for this endeavor. However, other labels could have been chosen; one could for instance stress that nomological reasoning is a black box type of reasoning, while causal reasoning attempts to replace the black box with a white box.

4. Problem 1: Bicycling

Our first illustrative problem is a case where nomological and causal reasonings work equally well. It reads:

> *How big is the force between foot and pedal compared to the friction force between road and tire when bicycling? Justify your answer.*

Making the drawing of figure 1 as a start, a solution to this problem based on *nomological* reasoning could look like this:

The cyclist is performing work on the pedal equal to a decrease of internal energy in her or him. The static friction force between road and tire is performing work equal to an increase in external energy (energy in the air due to air friction, potential energy when cycling uphill, or kinetic energy when accelerating). Due to conservation of energy the decrease in internal energy equals the increase in external energy. The work of the friction force is thus equal to the work of the pedal force.

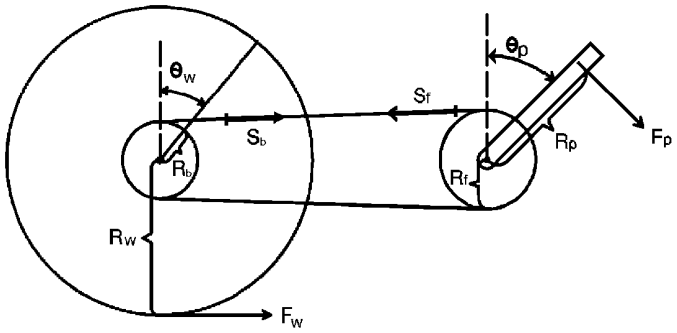


Fig 1 A diagram for the solution to problem 1.

Using that work is force times distance, we get that the equality of the two works can be expressed with the symbols in figure 1 as:

$$F_w R_w \Delta\theta_w = F_p R_p \Delta\theta_p \qquad (1)$$

Since the distance covered by the pedal gearwheel has to be equal to the distance covered by the wheel gearwheel, we get

$$R_b \Delta\theta_w = R_f \Delta\theta_p \qquad (2)$$

Hence the answer to the problem is:

$$\frac{F_p}{F_w} = \left(\frac{R_w}{R_p}\right) \cdot \left(\frac{R_f}{R_b}\right). \qquad (3)$$

This way of reasoning sees the phenomenon as governed by general laws, conservation of energy and the work-energy theorem. The reasoning does not appeal to any agent that causes an effect, in contrast to the following *causal* approach to the problem.

In this approach, we first look at the system consisting of the gear wheel in front, the pedal and a section of the chain. Since its angular momentum, and consequently its rate of change, is small, we consider the total torque on this system to be zero. Using the symbols on figure 1 we get:

$$R_f S_f = R_p F_p \text{ or } S_f = \left(\frac{R_p}{R_f}\right) F_p. \qquad (4)$$

Then we look at the part of the chain between S_f and S_b. Since the rate of change of its momentum is small because of its small mass, we have:

$$S_b = S_f \text{ or } S_b = \left(\frac{R_p}{R_f}\right) E_p. \qquad (5)$$

Finally, we look at the system consisting of the rear wheel, the rear gearwheel and a section of the chain. Again, we consider the total torque on the system to be zero:

$$R_w F_w = R_b S_b \text{ or } F_w = \left(\frac{R_b}{R_w}\right) S_b \qquad (6)$$

Combining equation 5 and equation 6 we again get equation 3 as our answer to problem 1.

This way of reasoning involves a number of causes in succession: First we have the foot pressing the pedal. This makes the front gearwheel turn. And that in turn tightens the chain. This then makes the rear gearwheel and the rear wheel turn. This finally gives rise to the static friction force between road and tire.

Both approaches apply the laws of physics to the situation, underlining that the difference between nomological and causal reasoning is related to *how* the general laws are used rather than *whether* these laws are used at all. The causal reasoning is more concrete than the nomological reasoning: First, it refers to specific causes and mechanisms that the problem solvers are familiar with from their everyday life. Second, the overall pattern used in the causal reasoning is more familiar from the everyday world where effects are often explained by a cause and not the general pattern of the nomological reasoning. The technical demands of the two approaches are roughly equivalent. Hence, both the nomological and the causal approaches lead to a solution of the problem, but they require different types of reasonings by the problem solver.

5. Problem 2: Puck collisions

Our second illustrative problem is a case where both nomological and causal reasonings are possible, like our first problem, but where the latter is much more involved than the former:

> *An ice hockey puck in motion on ice collides with a similar puck at rest. The pucks can be rotating before and after the collision, their rotational energies can change in the collision, and heat and internal oscillations can be produced within the pucks. How does the angle between the directions*

of motion of the two pucks after the collision depend on whether the total translational kinetic energy is increased, unchanged or decreased in the collision? Justify your answer.

Solving the problem with nomological reasoning, we sketch the situation as in figure 2, where $\mathbf{p}_{\text{initial}}$ is the initial momentum of the puck in motion before the collision, $\mathbf{p}_{\text{final1}}$ is the momentum of this puck after the collision, $\mathbf{p}_{\text{final2}}$ is the momentum of the other puck after the collision, and φ is the angle between the directions of the two pucks after the collision.

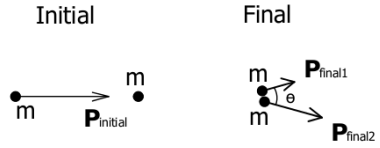


Fig. 2 A sketch of the collision of problem 2. The pucks are represented by their masses m.

Due to the absence of external horizontal forces acting on the system, the total momentum is conserved during the collision, that is

$$\mathbf{p}_{\text{initial}} = \mathbf{p}_{\text{final1}} + \mathbf{p}_{\text{final2}} \qquad (7)$$

By squaring this equation, we get

$$p_{initial}^2 = p_{final1}^2 + p_{final2}^2 + 2p_{final1} \cdot p_{final2} \cos\theta \quad (8)$$

If *m* is the mass of each puck, the total translational energy before and after the collision is, respectively, $\frac{p_{initial}^2}{2\text{m}}$ and $\frac{p_{final1}^2}{2\text{m}} + \frac{p_{final2}^2}{2m}$. So, equation (8) leads to the following conclusion: If the total translational energy increases during the collision, $\theta > 90^\circ$, if this energy decreases, $\theta < 90^\circ$ and if the energy is constant, $\theta = 90^\circ$.

This approach uses nomological reasoning because it shows how a general principle or law, conservation of momentum, determines the behavior of the system in the present situation. Taking a causal approach, we could analyze how the incoming puck collides with the other puck by studying how the two pucks interact during the collision via interaction forces. Such reasoning could help determine, for instance, how rotational energy of one puck is transferred to translational energy of the other puck by taking into account the frictional contact between the two pucks during the collision. Calculations along these lines are indeed possible (Hauge 2012) but they are much more technically demanding than the nomological approach. The nomological approach, however, does not give us the value of θ as a function of the initial situation.

Hence, this problem, too, can be approached nomologically as well as causally. The latter approach is much more technical demanding than the former but at the same time enables us to obtain additional information about e.g. the value of the angle between the final puck tracks.

6. Problem 3: Helium balloon

Our third illustrative problem is a case where, contrary to our previous problems, more is gained in the nomological approach than in the causal approach:

> *A child sitting in a bus holds a helium-filled balloon in a string. How does the direction of the balloon depend on how the bus is driving? Justify your answer.*

A solution to this problem based on causal reasoning could look like this: If the bus is at rest, the string of the balloon is vertical because the pressure is higher below the balloon than above the balloon. The situation is the same if the bus is driving with constant velocity. But when the bus accelerates the air inside the bus is compressed in the direction opposite to the acceleration. The air becomes denser at the opposite part of the bus relative to the direction of acceleration. Thus, the pressure is increased in that part of the bus and lowered in the part of the bus in the direction of the acceleration. Therefore, the direction of the string of the balloon is tilted from vertical towards the direction of the acceleration of the bus.

Here we explain the change of direction of the balloon string by a chain of causal mechanisms. Acceleration causes compression of air. Compression causes increased density. Increased density causes increased pressure. Finally, increased pressure causes the change of direction of the string.

A solution based on nomological reasoning could look like this: The string is having the direction of the buoyant force. The buoyant force is directed opposite to the resulting force field in the bus. If the bus is accelerating the resulting force field consists of the gravitional field, **g**, and the fictitious force field, **-a**, where **a** denotes the acceleration of the bus. Hence, the balloon will stand like in figure 3.

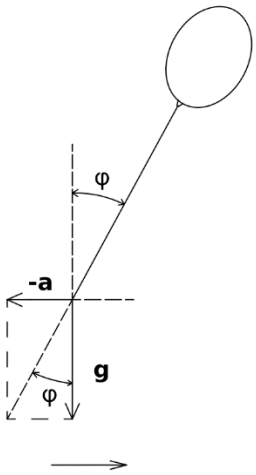


Fig. 3 A sketch of the balloon's position in problem 3.

Here the solution is based on the general pattern that phenomena in accelerated systems of reference acts like in inertial systems of reference if fictitious forces are considered. From figure 3 we can immediately calculate the angle ϕ that the string of the balloon forms with the normal of the bus floor:

$$\tan\phi = \frac{a}{g}. \qquad (9)$$

A similar calculation is not possible for the solution obtained in the causal approach. Here, it is difficult to transform the qualitative argument of this solution into a quantitative relation.

Hence, a solution to this problem can be found by using both nomological and causal reasoning, but the nomological approach gives us more information than the causal approach.

7. Problem 4: Flying

Our final problem illustrates that in some cases only a nomological approach is possible:

> *Why does an airplane tilt when changing its course? What is the relationship between the change of course and the tilting? Justify your answers.*

A solution using nomological reasoning could look like this: An airplane flying straight ahead is held up by a lift force perpendicular to the plane that is generated by the wings and the body of the airplane. The lift force is due to the distribution of pressure connected with the flow pattern of the air around the airplane. When the airplane tilts the flow pattern follows the airplane. Therefore, the lift force *L* tilts as well, i.e. it gets a non-vanishing horizontal component. And it is this horizontal component that brings about the change of course. Without tilting there is no horizontal component of the lift force and no change of course. That is why an airplane tilts when changing course.

If we assume the airplane is keeping its altitude while changing its course, the vertical component of the lift force must be opposite and numerically equal to the gravitational force on the airplane. If we denote the speed of the airplane by *v*, the radius of curvature of its trajectory *r* and its mass *m*, the horizontal component of the lift force must have the magnitude $\mathrm{m} \cdot \frac{\mathrm{v}^2}{\mathrm{r}}$.

From figure 2 it is then seen, that the connection between tilting (expressed by the angle ϕ) and changing of course (expressed by *r*) is:

$$\tan \phi = \frac{v^2}{r \cdot g}. \qquad (10)$$

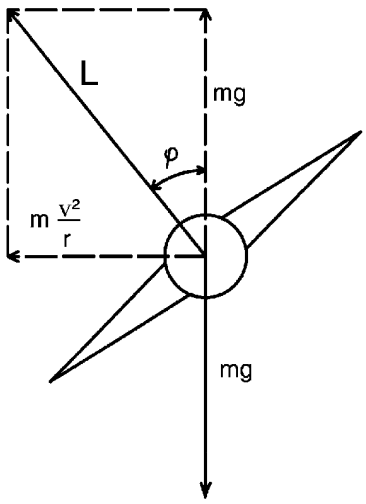


Fig. 4 A diagram of the flying problem

In this nomological approach, the problem solver applies a general pattern connecting circular motion and centripetal force. In contrast, a causal approach would argue in terms of causes and effects and perhaps hunt for a mechanism of how the pilot's turning of the control stick effects the wing flips that in turn effects the flow pattern of air that in turn causes the air plane to tilt. However, if at all feasible, such an argument requires the application of advanced aerodynamical knowledge rarely covered in physics programs.

The above solution is an instance of a general solution pattern shared by other problems such as *With what slope should a high way be constructed in a curve?* In more elaborate versions this problem is a standard problem in many textbooks. The optimal slope of a highway in a curve is the one forcing the car safe through the curve even when the road is covered with ice. In this case the friction force, in the plane of the

road and perpendicular to the driving direction of the car, is approximately zero. And in this case the changing of course of the airplane and of the car are completely analogous phenomena. Figure 2 shows the forces and their components when an airplane changes its course; this figure can be used also for the car in the curve of the road covered with ice just by replacing the lift force *L* on the figure with the normal reaction *N* from the road on the car. And the solution to the problem of which slope (expressed by the angle φ) a highway should be constructed in a curve (characterized by its radius of curvature *r*) is therefore also given by equation 10).

The analogy between the two problems is quite abstract and hence it is probably difficult for most students at an introductory college level to transfer their experiences from the highway problem to the airplane problem. The transfer of their experiences from the highway problem to an extension of the highway problem that asks for the friction force between the road and the car when the speed of the car is not adjusted to the slope of the highway, would, for instance, presumably be easier. This extended highway problem is technically more difficult than the airplane problem. But the extended problem deals with phenomenon and mechanisms in direct continuation of the original highway problem, which makes it easier to grasp intuitively. If, for instance, the car enters the curve with too low a speed most people will almost feel on their body that it will slide sidewise down the road, if it was not prevented by the mutual grasp of tire and road. And this friction force, as a reaction to the tendency of sliding, is also easy to imagine. Quite on the contrary no such supporting feelings exist when dealing with the airplane problem. Here you are left to the abstract reasoning of the physicist.

This problem shows that in some cases the nomological approach is the only feasible approach. However, the solution pattern is abstract and may be difficult to transfer to other situations.

8. Conclusion

The above problems illustrate that some problems – e.g. the airplane problem – can only be approached nomologically, while others can be approached both nomologically and causally – e.g. the first three problems. Moreover, we have also seen that for the problems of the latter kind, the nomological and the causal approaches can provide the same information (the bicycle problem), the causal approach can provide more information (the puck problem) or the nomological approach can provide more information (the helium balloon problem).

It is our experience as university physics teachers that when students solve problems they tend to cling to causal reasoning, which they prefer to nomological reasoning. The explanation is probably that causal reasoning is less abstract. Moreover, in our teaching we have observed that nomological reasoning demands more from the students than causal reasoning and that many students experience difficulties with nomological reasoning.

Even though physics uses both nomological and causal reasoning, it is important that teachers of physics are aware of the essential difference between the two types. And, since causal reasoning apparently is easier to digest and apply than nomological reasoning, it is important that the teachers discuss this difference with their students. But it would be a failure for physics teaching to avoid nomological reasoning just because this reasoning type is demanding: Partly because many problems, such as our airplane problem, can only be solved by taking a nomological approach. Partly because the subject of physics is in a particularly good position to demonstrate to students that such a thing as understanding based on nomological reasoning exists. It is in physics, above all, that understanding can be experienced not only as a question of finding mechanisms but also as a question of realizing the presence of governing patterns in

natural phenomena. And this insight influences both the cognitive development of the students as well as their interpretation of their own life and the surrounding world. However, teachers as well as students must be aware that it is perhaps the presence of nomological reasoning that makes physics a difficult subject.

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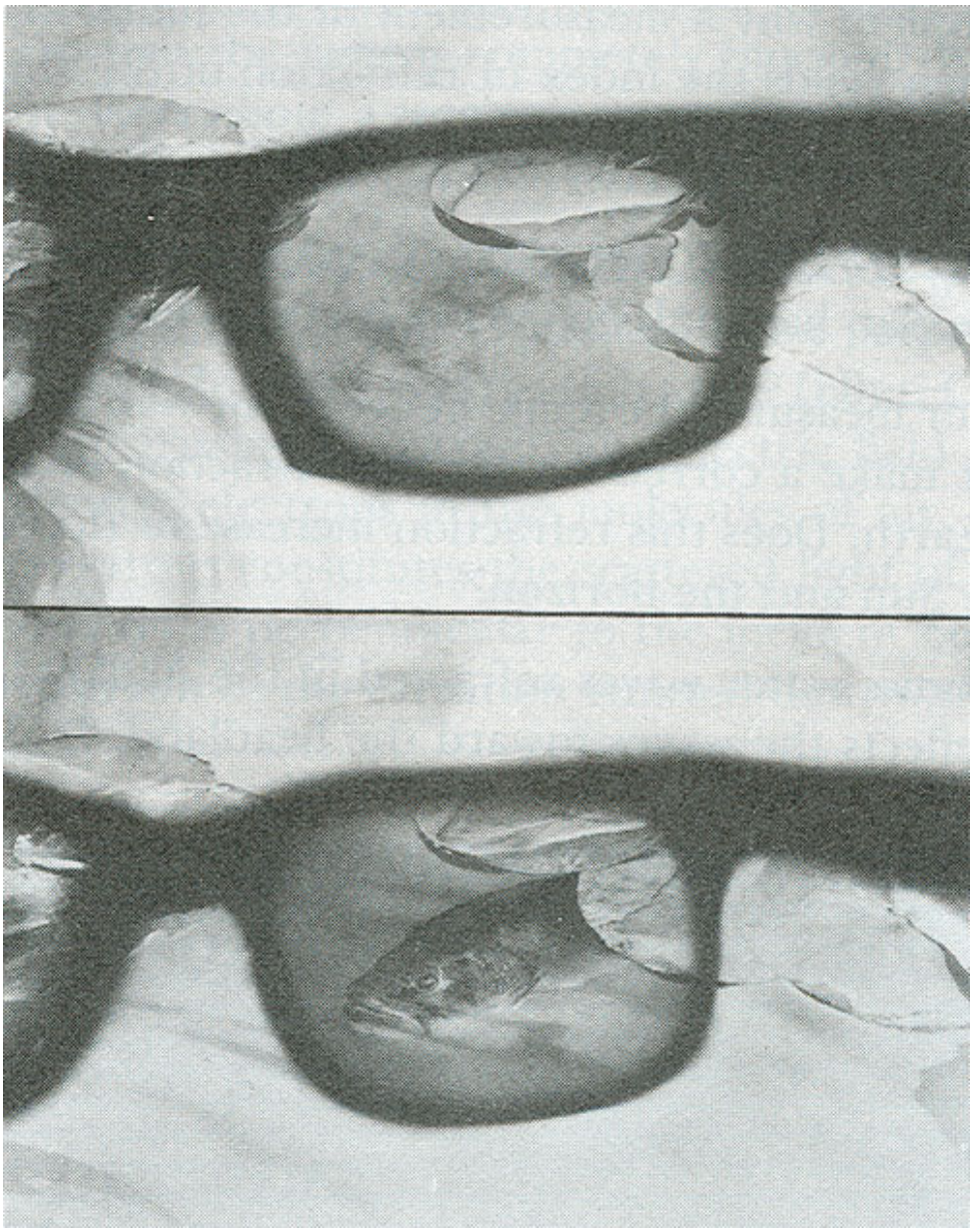

Fig. 37.32 When viewed through ordinary sunglasses (top), the fish is almost completely hidden by the glare reflected by the water. When viewed through Polaroid sunglasses (bottom), the fish becomes visible because the glare is reduced.

“Det sete afhænger af øjnene, der ser”