Criss-Cross Reduction of the Maslov Index and a Proof of the Yoshida-Nicolaescu Theorem

Bernhelm Booss-Bavnbek, Kenro Furutani, and Nobukazu Otsuki
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Introduction

As conjectured in [1], some spectral invariants of a closed partitioned manifold $M = M_0 \cup_\Sigma M_1$ with $M_0 \cap M_1 = \partial M_0 = \partial M_1 = \Sigma$ can be coded by the intersection geometry of the Cauchy data spaces along the partitioning hypersurface $\Sigma$ of associated differential operators. The Yoshida–Nicolaescu Formula belongs to this program. It was proved in [16] in dimension 3 and subsequently generalized in [12] and modified by several authors (see e.g. [2], [3], [7], [8]). It expresses the spectral flow of a family of Dirac operators with the same principal symbol but continuously varying connections by the Maslov intersection index of the Cauchy data spaces.

A closer look at the presently available spectral flow formulas shows that they differ quite a bit in the kind and strength of the underlying assumptions and the substance of the claims made. It seems to us that our framework (based on standard functional analysis involving only elementary distribution theory) permits a wider applicability than rather deep means (like the symbolic calculus and approximation theory).

First of all, the Cauchy data spaces are treated in slightly different ways. On one side, the Cauchy data spaces are established as $L^2$-closures of smooth


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sections over the partitioning hypersurface \( \Sigma \), coming from the restriction to the boundary of all smooth solutions over one of the parts \( M_j \) of the partitioned manifold \( M \). In the case of the Dirac operator, this Cauchy data space can be represented as the range of the \( L^2 \)-extension of the pseudo-differential Calderón projection and established as a Lagrangian subspace of the symplectic Hilbert space \( L^2(-\Sigma)+L^2(\Sigma) \) (for the somewhat delicate details see e.g. [4], pp. 75–104).

On the other side, the Cauchy data spaces can be established as subspaces of the symplectic Hilbert space \( \beta := D_{\max}/D_{\min} \) of natural boundary values, i.e. the boundary values of sections belonging to the maximal domain \( D_{\max} \) of the operator (so in [2], [3], see also [11]). One can embed \( \beta \) as a non-closed subspace into the distribution space \( H^{-1/2}(-\Sigma)+H^{-1/2}(\Sigma) \). This treatment of the Cauchy data spaces is independent of pseudo-differential analysis and has several conceptual and technical advantages: the arguments work for any symmetric elliptic differential operator of first order which satisfies a weak unique continuation property and admits an extension defining a self-adjoint Fredholm operator. Moreover, no product structures are required near the boundary. Finally, the space \( \beta \) is basic and most suitable for a short and general proof of the closedness and the Lagrangian property of the Cauchy data spaces and the continuity of the corresponding transformation from symmetric operators to Lagrangian subspaces. As observed already in [14], establishing this continuity is the crucial step in proving spectral flow formulas.

In [2], this \( \beta \)-program was carried out by standard functional analysis arguments and without assuming regularity at the endpoints or differentiability of the curve of operators, or other supplementary technical assumptions. The result was a quite general spectral flow formula, and the present paper is a continuation of [2]. We show, also by standard arguments, that an additional continuous transformation can be obtained, at least in the case when all metric structures are product near the separating (or non-separating) hypersurface, namely from the Cauchy data spaces of the \( \beta \)-theory into the ‘conventional’ \( L^2 \)-theory. This gives a slight generalization and a new proof of the Yoshida–Nicolaescu Formula.

In Section 1 we address a fairly general situation in symplectic functional analysis and prove a ‘criss–cross’ reduction theorem for the Maslov index. In Section 2 we give a condensed version of the General Spectral Flow Formula of [2]. In Section 3 we connect the two preceding sections and give a new proof of the Yoshida–Nicolaescu Formula. In the Appendix we correct an erroneous description in [2], §§ 1.1-1.2, and add a lemma to be inserted at the end of § 1.2 in [2].
1. Criss-Cross Reduction of the Maslov Index

Let $\beta$ and $L$ be symplectic Hilbert spaces with symplectic forms $\omega_\beta$ and $\omega_L$, respectively. Let

$$\beta = \beta_- \oplus \beta_+ \quad \text{and} \quad L = L_- \oplus L_+$$

be direct sum decompositions by transversal (not necessarily orthogonal) pairs of Lagrangian subspaces. We assume that there exist continuous, injective mappings

$$i_- : \beta_- \to L_- \quad \text{and} \quad i_+ : L_+ \to \beta_+$$

with dense images and which are compatible with the symplectic structures, i.e.

$$\omega_L(i_-(x), a) = \omega_\beta(x, i_+(a)) \quad \text{for all } a \in L_+ \text{ and } x \in \beta_-.$$  

Let $\lambda_0$ be a fixed Lagrangian subspace of $\beta$. We denote the Fredholm Lagrangian Grassmannian of $\lambda_0$ by $\mathcal{FL}_{\lambda_0}(\beta)$. It is the space of all Lagrangian subspaces of $\beta$ which make a Fredholm pair with $\lambda_0$. Its topology is defined by the operator norm of the projections. Its fundamental group is $\mathbb{Z}$, and the mapping of the loops in $\mathcal{FL}_{\lambda_0}(\beta)$ onto $\mathbb{Z}$ is given by the Maslov index. It is an intersection index of the loop with the Maslov cycle

$$\mathcal{M}_{\lambda_0}(\beta) := \{ \lambda \in \mathcal{FL}_{\lambda_0}(\beta) \mid \lambda \cap \lambda_0 \neq \{0\} \}.$$  

By functional analytical means, the definition of the Maslov index can be extended unambiguously to all continuous curves in $\mathcal{FL}_{\lambda_0}(\beta)$ (see [2] and [3], inspired by [15]; we refer to [6] for various aspects of the Maslov index and also to [9], [10] for a cohomological treatment).

We are going to prove the following

**Theorem 1.1.** Under the assumptions (1.1), (1.2), and (1.3),

(a) we have a continuous mapping

$$\tau : \mathcal{FL}_{\beta_-}(\beta) \to \mathcal{FL}_{L_-}(L)$$

(b) which maps the Maslov cycle of $\beta_-$ into the Maslov cycle of $L_-$ and

(c) preserves the Maslov index

$$\text{mas} \left( \{ \lambda_t \}_{t \in [0,1]}, \beta_- \right) = \text{mas} \left( \{ \tau(\lambda_t) \}_{t \in [0,1]}, L_- \right)$$

for any continuous curve $[0,1] \ni t \mapsto \lambda_t \in \mathcal{FL}_{\beta_-}(\beta)$.

We prove Theorem 1.1 in a series of small lemmata.

**1.1. Definition of the mapping $\tau$.** We consider the direct sum

$$\mathcal{D} := \beta_+ \oplus L_-,$$

where $\beta$ and $L$ are identified with subspaces of $\mathcal{D}$. Then we define the mapping $\tau$ simply by

$$\tau(\lambda) := \lambda \cap L \quad \text{for } \lambda \in \mathcal{FL}_{\beta_-}(\beta).$$
Clearly
\begin{equation}
\tau(\lambda) = \{i_-(a) + x \mid a \in \beta_-, x \in L_+ \text{ such that } i_+(x) + a \in \lambda\}.
\end{equation}

To prove that range(\tau) \subset \mathcal{FL}_{L_-}(L) and that \tau is continuous, we introduce an alternative description of \tau(\lambda) in terms of bounded operators associated to \lambda. For a given \lambda \in \mathcal{FL}_{\beta_-}(\beta), we fix a direct sum decomposition
\[\lambda = (\lambda \cap \beta_-) \oplus \mu\]
with a suitable closed \mu. Let \pi_+ : \beta \to \beta_+ denote the projection along \beta_. Then to claim that (\lambda, \beta_-) is a Fredholm pair is equivalent to claiming that the projection \pi_+ := \pi_+|_{\lambda} : \lambda \to \beta_+ is a Fredholm operator. So we deduce that \[F_\lambda := \pi_+(\lambda) = \pi_+(\mu)\] is closed. For later use we notice that
\begin{equation}
\dim \beta_+/F_\lambda = \dim(\beta_+ + \beta_-) / (\lambda + \beta_-) = \dim \lambda \cap \beta_- < +\infty.
\end{equation}

By the injectivity of \pi_+|_{\mu} we can write \mu as the graph of a uniquely determined bounded operator
\[f_\mu : F_\lambda \to \beta_.\]

Then we rewrite
\begin{equation}
\tau(\lambda) = i_-(\lambda \cap \beta_-) + \text{graph}(\varphi_\lambda),
\end{equation}
where
\[\varphi_\lambda : i_+^{-1}(F_\lambda) \to L_-
\]
\[x \mapsto i_- \circ f_\mu \circ i_+(x)\]

Since \varphi_\lambda is bounded and its domain is closed in \lambda, its graph is also closed in \lambda, and so is \tau(\lambda) which differs from graph(\varphi_\lambda) only by a space of finite dimension. But moreover:

**Lemma 1.2.** For each \lambda \in \mathcal{FL}_{\beta_-}(\beta), we have \tau(\lambda) \in \mathcal{FL}_{L_-}(L).

**Proof.** In the exact sequence
\[
0 \to F_\lambda \to \beta_+ \overset{p}{\to} \beta_+/F_\lambda \to 0
\]
the range of \(i_+\) is dense in \beta_+ and \(\dim \beta_+/F_\lambda < +\infty\). So, the map \(p \circ i_+\) is surjective. We therefore have
\begin{equation}
\beta_+ = F_\lambda + i_+(L_+)
\end{equation}
and \(\ker(p \circ i_+) = i_+^{-1}(F_\lambda)\). It follows that
\begin{equation}
L_+/i_+^{-1}(F_\lambda) \cong (F_\lambda + i_+(L_+))/F_\lambda = \beta_+/F_\lambda.
\end{equation}

This implies that the closed subspace \(i_+^{-1}(F_\lambda)\) of \(L_+\) is of finite codimension.
Now let \( x, y \in i_+^{-1}(F_\lambda) \) and set \( a := i_+(x) \) and \( b := i_+(y) \). Then
\[
\omega_L(x + \varphi_\lambda(x), y + \varphi_\lambda(y)) = \omega_L(x, \varphi_\lambda(y)) + \omega_L(\varphi_\lambda(x), y) \\
= \omega_\beta(a, f_\mu(b)) + \omega_\beta(f_\mu(a), b) \\
= \omega_\beta(a + f_\mu(a), b + f_\mu(b)) = 0
\]
by the compatibility condition (1.3) and the isotropy of \( \mu \). So graph \( \varphi_\lambda \) is isotropic. Moreover, we have
\[
\dim(\text{graph}(\varphi_\lambda)^0 / \text{graph}(\varphi_\lambda)) = 2 \dim(L_+ / i_+^{-1}(F_\lambda)) = 2 \dim(\beta_+/F_\lambda).
\]
Here \( \text{graph}(\varphi_\lambda)^0 \) denotes the annihilator of graph(\( \varphi_\lambda \)) with respect to the symplectic form \( \omega_L \). Clearly,
\[
i_- (\lambda \cap \beta_-) \subset \text{graph}(\varphi_\lambda)^0 \quad \text{and} \quad i_- (\lambda \cap \beta_-) \cap \text{graph}(\varphi_\lambda) = \{0\}.
\]
Further, we have
\[
\dim i_- (\lambda \cap \beta_-) = \dim(\lambda \cap \beta_-) = \dim(\beta_+/F_\lambda),
\]
since the Fredholm index of the Lagrangian pair \( (\lambda, \beta_-) \) vanishes. From this dimension examination we see that adding \( i_- (\lambda \cap \beta_-) \) to graph(\( \varphi_\lambda \)) makes \( \tau(\lambda) \) a Lagrangian subspace of \( L \). It also follows that \( \tau(\lambda) \cap L_- = i_- (\lambda \cap \beta_-) \) is of finite dimension and that \( L_- + \tau(\lambda) = L_- + \text{graph}(\varphi_\lambda) \) is of finite codimension, hence \( (\tau(\lambda), L_-) \) is a Fredholm pair in \( L \). \( \square \)

**Remark 1.3.** From equation (1.7) it particularly follows that
\[
\tau(M_{\beta_-}(\beta)) \subset M_{L_-}(L).
\]

1.2. The continuity of \( \tau \). We fix a closed \( W \subset L_- \) with \( \dim L_- / W < +\infty \) and choose a Lagrangian subspace \( \theta \) of \( \beta \) with \( \theta \cap i_+^{-1}(W) \) and \( W_{\beta} \subset \theta \subset W_{\beta}^0 \), where \( W_{\beta} := i_+^{-1}(W) \). Here "\( \cap \)" means that the two subspaces intersect transversally. We notice that
\[
W^0 = L_- + (L_+ \cap W^0)
\]
and, correspondingly,
\[
W_{\beta}^0 = \beta_+ + (\beta_+ \cap W_{\beta}^0).
\]

Next we exploit that the injection \( i_- : \beta_- \rightarrow L_- \) has a dense range. Similar to the short exact sequence at the beginning of the proof of Lemma 1.2, we have
\[
0 \rightarrow W \hookrightarrow L_- \overset{\varphi}{\rightarrow} L_- / W \rightarrow 0
\]
\[
\uparrow i_- \quad \beta_- \quad \beta_{\beta}
\]
We deduce, as above,
\[
L_- = W + i_-(\beta_-) \quad \text{and} \quad \dim(\beta_- / W_{\beta}) = \dim(L_- / W).
\]
Moreover, we have
\[
i_+(W^0 \cap L_+) = W_{\beta}^0 \cap \beta_+.
\]
\[1.10\]
To show the inclusion \( \subset \), we consider any \( x \in W^0 \cap L_+ \). Then \( \omega_L(x, y) = 0 \) for all \( y \in W \), and hence also for \( y = i_-(z) \) for any \( z \in W_\beta = i_-^{-1}(W) \). Now, by (1.3),
\[
0 = \omega_L(x, i_-(z)) = \omega_\beta(i_+(x), z).
\]
So,
\[
i_+(x) \in W_\beta^0 \cap \beta_+.
\]
We deduce the identity \( \subset \) by the following dimension examination:
\[
2 \dim(W^0 \cap L_+) = \dim(W^0/W) = \dim(W_\beta^0/W_\beta) = 2 \dim(W_\beta^0 \cap \beta_+).
\]
Equation (1.10) permits us to rewrite
\[
W_\beta^0 = i_+(L_+ \cap W^0) + \beta_-
\]
and to define
\[
i_W : W_\beta^0 \longrightarrow W^0
\]
\[
i_+(x) + z \mapsto x + i_-(z)
\]
for \( x \in L_+ \cap W^0 \) and \( z \in \beta_- \). We obtain a new splitting of the symplectic Hilbert spaces:

**Proposition 1.4.** The space \( \eta := i_W(\theta) + W \) is a Lagrangian subspace of \( L \) and the mapping \( i_W|_\theta : \theta \to \eta \) has a dense image. Further, we have new direct sum decompositions
\[
\beta = \beta_+ + \theta \quad \text{and} \quad L = L_+ + \eta
\]
which are compatible with regard to the symplectic forms \( \omega_L \) and \( \omega_\beta \) (similar to (1.3)).

**Proof.** Clearly
\[
W \subset \eta = i_W(\theta) + W \subset W^0 \quad \text{and} \quad \dim W^0/W < +\infty,
\]
hence \( \eta \) is also closed and isotropic. We notice that the mapping \( q \circ i_W|_\theta \) is surjective with \( \ker q \circ i_W|_\theta = W_\beta \), where \( q : \eta \to \eta/W \) denotes the projection. So,
\[
W_\beta^0/W_\beta \supset \theta/W_\beta \cong \eta/W \subset W^0/W.
\]
Then, from the dimension examination
\[
\dim(W_\beta^0/W_\beta) = \dim(W^0/W),
\]
it follows that \( \eta \) is a Lagrangian subspace of \( L \).

To see that the range of \( i_W|_\theta \) is dense in \( \eta \) we recall that the mapping
\[
i_W : W_\beta^0 \to W^0
\]
is an isomorphism \( W_\beta^0 \cap \beta_+ \cong W^0 \cap L_+ \) on the first finite-dimensional component, and it is equal to the dense embedding \( i_- : \beta_- \to L_- \) on the second component, hence its restriction to \( \theta \) has a dense range in \( \eta \).

The new direct sum decompositions and the compatibility of the symplectic forms follow at once. \( \square \)

The preceding proposition permits a further simplification of the graph representation of \( \tau(\lambda) \), obtained in (1.7):
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COROLLARY 1.5. For any \( \lambda \in \mathcal{F}\mathcal{L}_{\beta_-}(\beta) \) with \( \lambda \pitchfork \theta \), we have

\[
\tau(\lambda) = \text{graph}(i_W \circ f_\lambda \circ i_+) ,
\]

where \( f_\lambda : \beta_+ \to \theta \) such that \( \text{graph} f_\lambda = \lambda \).

Now we can prove the continuity of \( \tau \) and the invariance of the Maslov index under \( \tau \). We set

\[
\mathcal{F}\mathcal{L}_{W_\beta}(\beta)^{(0)} := \{ \mu \in \mathcal{F}\mathcal{L}_{W_\beta}(\beta) \mid \mu \cap W_\beta = \{0\} \}.
\]

Then we have an open covering

\[
\bigcup_{W} \mathcal{F}\mathcal{L}_{W_\beta}(\beta)^{(0)} = \mathcal{F}\mathcal{L}_{\beta_-}(\beta) ,
\]

where the union is taken over all closed subspaces \( W \subset L_- \) of finite codimension. It follows that \( \tau(\mathcal{F}\mathcal{L}_{W_\beta}(\beta)^{(0)}) \subset \mathcal{F}\mathcal{L}_W(L)^{(0)} \). Thus, we have established restrictions

\[
\tau_W : \mathcal{F}\mathcal{L}_{W_\beta}(\beta)^{(0)} \to \mathcal{F}\mathcal{L}_W(L)^{(0)}
\]

of \( \tau \) for each \( W \).

We fix a \( W \). We denote the space of Lagrangian subspaces in a finite-dimensional symplectic space by \( \text{Lag}(\cdot) \) and define the reduction map

\[
\rho_{W_\beta} : \mathcal{F}\mathcal{L}_{W_\beta}(\beta)^{(0)} \to \text{Lag}(W^0_\beta / W_\beta) ;
\lambda \mapsto (\lambda \cap W^0_\beta + W_\beta) / W_\beta.
\]

Then, for each Lagrangian subspace \( \theta \) in \( \beta \) with \( \theta \supset W_\beta \) and \( \theta \pitchfork \beta_+ \) (as before), the set

\[
U(W_\beta, \theta) := \{ \Lambda \in \text{Lag}(W^0_\beta / W_\beta) \mid \Lambda \cap \theta / W_\beta \}
\]

is open in \( \text{Lag}(W^0_\beta / W_\beta) \) and we have an open covering

\[
\bigcup_{\theta \pitchfork \beta_+} \rho_{W_\beta}^{-1}(U(W_\beta, \theta)) = \mathcal{F}\mathcal{L}_{W_\beta}(\beta)^{(0)} .
\]

By Corollary 1.5, the mapping \( \tau \) is continuous on each \( \rho_{W_\beta}^{-1}(U(W_\beta, \theta)) \), and so it is continuous on the whole \( \mathcal{F}\mathcal{L}_{\beta_-}(\beta) \). This concludes the proof of (a) of Theorem 1.1.

Moreover, we have

\[
\tau(\rho_{W_\beta}^{-1}(U(W_\beta, \theta))) \subset \rho_{W}^{-1}(U(W, \tau_W(\theta))) ,
\]

where the reduction map \( \rho_W : \mathcal{F}\mathcal{L}_W(L)^{(0)} \to \text{Lag}(W^0 / W) \) is defined correspondingly to \( \rho_{W_\beta} \). We denote by

\[
\overline{\tau}_W : \text{Lag}(W^0_\beta / W_\beta) \to \text{Lag}(W^0 / W)
\]

the mapping naturally induced by \( \tau_W \). Then we have the following finite-dimensional symplectic reduction:
Proposition 1.6. The diagram
\[
\begin{array}{ccc}
\mathcal{F}\mathcal{L}_{\beta}(\beta) & & \mathcal{F}\mathcal{L}_{W}(\beta)^{(0)} \\
\uparrow\tau & & \downarrow\tau_{W} & \xrightarrow{\rho_{W}} \text{Lag}(W_{\beta}/W_{\beta}) \\
\mathcal{F}\mathcal{L}_{L}(L) & & \mathcal{F}\mathcal{L}_{W}(L)^{(0)} & \xrightarrow{\rho_{W}} \text{Lag}(W^{0}/W)
\end{array}
\]
(1.11)
is commutative.

For any continuous curve \(\{\lambda_{t}\}_{t \in [0,1]} \in \mathcal{F}\mathcal{L}_{\beta}(\beta)\), we can find a closed subspace \(W \subset L_{-}\) of finite codimension such that the whole curve is contained in \(\mathcal{F}\mathcal{L}_{W}(\beta)^{(0)}\), respectively \(\{\tau(\lambda_{t})\}_{t \in [0,1]} \) is contained in \(\mathcal{F}\mathcal{L}_{W}(L)^{(0)}\). By finite symplectic reduction we obtain at once:

Corollary 1.7. The Maslov index coincides under the transformations \(\tau\), \(\tau_{W}\), and \(\tau_{W}\) for loops and also for paths.

Note. For paths, the Maslov index depends on the choice of the Maslov cycle. In \(\text{Lag}(W^{0}/W)\), the Maslov index is taken with respect to the Maslov cycle \(\mathcal{M}_{L_{-}/W}(W^{0}/W)\). Correspondingly, the Maslov index in \(\text{Lag}(W_{\beta}^{0}/W_{\beta})\) is taken with respect to the Maslov cycle \(\mathcal{M}_{\beta_{-}/W}(W_{\beta}^{0}/W_{\beta})\). Note that the specified Maslov cycle in \(\text{Lag}(W_{\beta}^{0}/W_{\beta})\) is mapped onto the specified Maslov cycle in \(\text{Lag}(W^{0}/W)\) by \(\tau_{W}\).

Proposition 1.6 and Corollary 1.7 together with Remark 1.3 give the proof of (b) and (c) of Theorem 1.1.

2. The General Spectral Flow Formula

Let \(H\) be a real separable Hilbert space and \(A\) an (unbounded) closed symmetric operator defined on the domain \(D_{\min}\) which is supposed to be dense in \(H\). Let \(A^{*}\) denote its adjoint operator with domain \(D_{\max}\). We have that \(A^{*}|_{D_{\min}} = A\) and that \(A^{*}\) is the maximal closed extension of \(A\) in \(H\).

We form the space \(\beta\) of natural boundary values with the natural trace map \(\gamma\) in the following way:
\[
\begin{array}{c}
D_{\max} \\
x
\end{array}
\xrightarrow{\gamma}
\begin{array}{c}
D_{\max}/D_{\min} =: \beta \\
\gamma(x) = [x] := x + D_{\min}
\end{array}.
\]
The space \(\beta\) becomes a symplectic Hilbert space with the scalar product induced by the graph norm
\[
(x, y)_{\beta} := (x, y) + (Ax, Ay)
\]
and the symplectic form given by Green's form
\[
\omega([x], [y]) := (Ax, y) - (x, Ay) \quad \text{for } [x], [y] \in \beta.
\]
We define the natural Cauchy data space \(\Lambda := \gamma(\ker A^{*})\). It is a Lagrangian subspace of \(\beta\) under the assumption that \(A\) admits at least one self-adjoint Fredholm extension \(D_{A}\). Actually, we shall assume a little more, namely that
A has a self-adjoint extension $A_D$ with compact resolvent. Then $(\Lambda, \gamma(D))$ is a Fredholm pair of subspaces of $\beta$, i.e. $\Lambda \in FL_c(D)(\beta)$.

We consider a continuous curve $\{C_t\}_{t \in [0,1]}$ in the space of bounded self-adjoint operators on $\mathcal{H}$ and assume that the operators $A^* + C_t - \tau$ have no 'inner solutions', i.e. satisfy the weak unique continuation property (UCP)

$$\ker(A^* + C_t - \tau) \cap D_{\min} = \{0\}$$

for $t \in [0,1]$ and $|\tau| < \varepsilon_0$ with $\varepsilon_0 > 0$.

Clearly, the domains $D_{\max}$ and $D_{\min}$ are unchanged by the perturbation $C_t$ for any $t$. So, $\beta$ does not depend on the parameter $t$. Moreover, the symplectic form $\omega$ is invariantly defined on $\beta$ and so also independent of $t$. It follows (see [2], Theorem 3.9) that the curve $\{\Lambda_t := \gamma(\ker(A^* + C_t))\}$ is continuous in $FL_c(D)(\beta)$.

Given this, the family $\{A_D + C_t\}$ can be considered at the same time in the spectral theory of self-adjoint operators, defining a spectral flow, and in the symplectic category, defining a Maslov index. Under the preceding assumptions, the main result obtainable at that level is the following general spectral flow formula (proved in [2], Theorem 5.1):

**Theorem 2.1.** Let $A_D$ be a self-adjoint extension of $A$ with compact resolvent and let $\{A_D + C_t\}$ be a family satisfying the UCP assumption. Then

$$\text{sf}\{A_D + C_t\} = \text{mas}\left(\{\Lambda_t\}, \gamma(D)\right).$$

3. A Proof of the Yoshida–Nicolaescu Formula

Let $M$ be a closed connected smooth Riemannian manifold and let $\Sigma \subset M$ be a hypersurface. Let

$$\{A_t := A_0 + C_t : C^\infty(M; S) \to C^\infty(M; S)\}_{0 \leq t \leq 1}$$

be a continuous family of symmetric elliptic differential operators of first order with the same principal symbol, acting on sections of a real bundle $S$ over $M$. The variation of the fixed operator $A_0$ is given by a continuous family $\{C_t\}_{t \in [0,1]}$ of smooth symmetric bundle homomorphisms. We assume that the normal bundle of the hypersurface $\Sigma$ is orientable and that all metric structures of $M$ and $S$ are product in a collar neighbourhood $N = (-1,1) \times \Sigma$ of $\Sigma$. We have

$$A_t = \sigma(\frac{\partial}{\partial t} + B_t) \quad \text{on} \quad N,$$

where $t$ denotes the normal coordinate, $\sigma$ is orthogonal (assuming $\sigma^2 = -1\text{Id}$), $\sigma B_t = -B_t \sigma$, and $B_t$ is a self-adjoint elliptic differential operator over $\Sigma$, called the tangential operator. Here the point of the product structure is that then $\sigma$ and $B_t$ do not depend on the normal variable.

We cut the manifold at $\Sigma$ and attach a copy of $\Sigma$ to each side. So, we obtain a new manifold $M_t$ with boundary $\Sigma_0 \cup \Sigma_1 = (\Sigma) \cup \Sigma$. Then the $\beta$-space of
\( M_4 \), being a \( C^\infty(\partial M_4) \)-module, splits according to the connected components of \( \partial M_4 \) and we obtain two symplectic Hilbert spaces

\[
\beta := \beta_0 \dot{+} \beta_1 \quad \text{and} \quad L := L_0 \dot{+} L_1
\]

with

\[
\gamma = \gamma_0 \oplus \gamma_1 : D_{\max}(M_4) \to \beta_0 \dot{+} \beta_1
\]

and \( L_j = L^2(\pm \Sigma; S|_{\pm \Sigma}) \) for \( j = 0, 1 \). We note that \( L_0 \) and \( L_1 \) are naturally identified as Hilbert spaces, but their symplectic forms have opposite signs. In \( \beta \) we have a Lagrangian subspace

\[
\Delta_\beta := \{(x, x) \in \beta \dot{+} \beta_1 \mid \gamma_0(s) = x = \gamma_1(s) \text{ for } s \in H^1(M)\},
\]

where \( H^1(M) \) denotes the first order Sobolev space. Since \( M \) is closed, the operator \( A_\ell \) on \( H^1(M) \) is a self-adjoint Fredholm operator. It can be identified with the operator \( A^\delta_{1,D} = A^\delta_{0,D} + C_\ell \) over the new manifold \( M_4 \) with domain

\[
D := \{ s \in D_{\max}(M_4) \mid \gamma_0(s) = \gamma_1(s) \},
\]

i.e. it can be considered as a global self-adjoint boundary problem with \( \gamma(D) = \Delta_\beta \). (Actually, \( D \) coincides with \( H^1(M) \)).

We set \( \beta_- := \Delta_\beta \). In \( L \), we have a Lagrangian subspace

\[
L_- := \text{ the diagonal of } L^2(\Sigma; S|_{\Sigma}) \dot{+} L^2(\Sigma; S|_{\Sigma}).
\]

Clearly, the embedding \( \beta_- \hookrightarrow L_- \) is bounded and dense.

To define suitable complementary Lagrangian subspaces \( \beta_\pm \) and \( L_\pm \), we must fall back on a spectral resolution \( \{ \varphi_k, \lambda_k \} \) of \( L^2(\Sigma) \) by eigensections of \( B_0 \).

(Here and in the following we do not mention the bundle \( S \)). For simplicity, we assume \( \ker B_0 = \{ 0 \} \). Otherwise we must decompose the finite-dimensional symplectic vector space \( \ker B_0 \) into two Lagrangian subspaces and add these spaces to the half parts defined by the spectral cut at 0.

Following [3], Proposition 7.15 (for related results see also [5] and [11]), we decompose

\[
\beta_0 = \beta_-^0 \dot{+} \beta_+^0 \quad \text{and} \quad \beta_1 = \beta_-^1 \dot{+} \beta_+^1,
\]

where

\[
\beta_-^1 := \{(\varphi_k)_{k < 0}\}_{H^1(\Sigma)} \quad \text{and} \quad \beta_-^1 := \{(\varphi_k)_{k > 0}\}_{H^{-1/2}(\Sigma)}
\]

and

\[
\beta_-^0 := \{(\varphi_k)_{k < 0}\}_{H^{-1/2}(\Sigma)} \quad \text{and} \quad \beta_-^0 := \{(\varphi_k)_{k > 0}\}_{H^{1/2}(\Sigma)}.
\]

In a similar way, we decompose

\[
L_0 = L_-^0 \dot{+} L_+^0 \quad \text{and} \quad L_1 = L_-^1 \dot{+} L_+^1
\]

with

\[
L_-^1 := \{(\varphi_k)_{k < 0}\}_{L^2(\Sigma)} \quad \text{and} \quad L_+^1 := \{(\varphi_k)_{k > 0}\}_{L^2(\Sigma)}
\]
and
\[ L_0 := \{ \{ \varphi_k \}_{k < 0} \}_{L^2(\Sigma)} \quad \text{and} \quad L^0_+ := \{ \{ \varphi_k \}_{k > 0} \}_{L^2(\Sigma)}. \]

Rewriting
\[ \beta = \beta_0 + \beta_1 = \beta_0^- + \beta_1^- + \beta_1^+, \]
we obtain
\[ \beta_- = \Delta_\beta = \{ (a, b, a, b) \mid (a, b) = \gamma_0(s) = \gamma_1(s) \text{ with } s \in H^1(M) \}. \]
Correspondingly, we define
\[ \beta_+ := \{ (x, 0, 0, y) \mid x \in \beta_0^-, y \in \beta_1^+ \} \]
and
\[ L_+ := \{ (u, 0, 0, v) \mid u \in L_0^-, v \in L_1^+ \}. \]

Clearly, we have \( L = L_- + L_+ \), a dense bounded embedding \( L_+ \hookrightarrow \beta_+ \), and \( \Delta_\beta \cap \beta_+ = \{ 0 \} \). From the decomposition (3.4) we have also that \( \beta_- + \beta_+ = \beta \).

Then, as explained above in Section 2, the \( \beta \)-theory gives the continuity of the family of Cauchy data spaces \( \{ \Lambda_t \} \) (of the continuous operator family \( \{ A_t \} \), considered over \( M_t \)). They are all Lagrangian subspaces of \( \beta \) and make Fredholm pairs with \( \beta_- = \Delta_\beta = \gamma(H^1(M)) \).

From the General Spectral Flow Formula (here Theorem 2.1) and by the criss–cross reduction of the Maslov index (Theorem 1.1), we obtain
\[ \text{sf} \{ A_0 + C_t \} = \text{sf} \{ A_t^0 + C_t \} \]
\[ \overset{\text{Th} 2.1}{=} \text{mas} (\{ \Lambda_t \}, \Delta_\beta) \]
\[ \overset{\text{Th} 1.1}{=} \text{mas} (\{ \Lambda_t \cap L \}, L_-). \]

Note that here
\[ \Lambda_t \cap L = \{ s|_{(-\Sigma)\cup \Sigma} \mid s \in H^\frac{1}{2}(M_t) \text{ and } A_t(s) = 0 \text{ in } M_t \setminus \partial M_t \}, \]
i.e. it coincides with the ‘conventional’ \( L^2 \)-definition of the Cauchy data spaces.

In the particular case of a partitioned manifold
\[ M = M_0 \cup \Sigma M_1 \text{ with } \Sigma = \partial M_0 = \partial M_1 = M_0 \cap M_1, \]
not only \( \beta \) and \( L \) split but also the Cauchy data spaces split
\[ \Lambda_t = \Lambda_t^0 + \Lambda_t^1, \]
according to the splitting of \( M \) into two parts. So, we obtain from (3.7) the true Yoshida–Nicolaescu Formula, though without any assumptions about the regularity at the endpoints or the differentiability of the curve:

**THEOREM 3.1.**
\[ \text{sf} \{ A_0 + C_t \} = \text{mas} (\{ \Lambda_t^0 \cap L^2(-\Sigma) + \Lambda_t^1 \cap L^2(\Sigma) \}, L_-) \]
\[ =: \text{mas} (\{ \Lambda_t^0 \cap L^2(-\Sigma) \}, \{ \Lambda_t^1 \cap L^2(\Sigma) \}), \]
where the last expression is just a rewriting of the Maslov index of Fredholm pairs of two curves.
Remark 3.2. (a) We notice that the subspaces $\beta_-$ and $L_-$ are defined independently of a reference tangential operator, here $B_0$, but the choice of any other reference operator $B_t$ would have given a different decomposition, though not a different result. 
(b) We would like to emphasize that the Yoshida–Nicolaeescu formulas in the literature always assume product structures near $\Sigma$, whereas the general spectral flow formulas, as proved in [2] and expressed in [3] in $\beta \subset H^{-1/2}(\Sigma)$, do not require product structures near $\Sigma$. However, we also need product structures near $\Sigma$, to apply our criss-cross reduction theorem and to transform the general spectral flow formulas, expressed in ‘natural’ distribution spaces, into ‘conventional’ $L^2$–formulas. Possibly, the criss-cross reduction theorem may provide one explanation for the need of product structures for $L^2$–formulas.
(c) We also want to point to a misprint in [3], p. 74 (after Equation (7.13)), where it must read that ‘$\gamma(S) \cap L^2(\Sigma)$ is not closed in $L^2(\Sigma)$’ instead of ‘$\gamma(S) \cap L^2(\Sigma)$ is not closed in $L^2(\Sigma)$’.

Appendix A. Addendum and Corrections to [2]

Let $L$ denote the space of all Lagrangian subspaces of a real separable symplectic Hilbert space $\mathcal{H}$, and let $L^C$ denote the space of all complex Lagrangian subspaces of $\mathcal{H} \otimes \mathbb{C}$. As correctly observed in [2], the full group $U(\mathcal{H})$ of unitary operators of $\mathcal{H}$ does not act on $\mathcal{F}\Lambda^0(\mathcal{H})$, but the reduced group $U_c(\mathcal{H})$ does. It consists of unitary operators of the form $\text{Id} + K$, where $K$ is a compact operator. In [2], below on p. 5, we claimed that the reduced group acts transitively on $\mathcal{F}\Lambda^0(\mathcal{H})$, i.e. the mapping

$$\rho : U_c(\mathcal{H}) \rightarrow \mathcal{F}\Lambda^0(\mathcal{H})$$

$$U \mapsto U(\lambda^0)$$

is surjective. This claim is erroneous, since the difference of the orthogonal projections onto a Fredholm pair of closed subspaces of $\mathcal{H}$ need not be of the form $\text{Id} + K$, but can become an arbitrary Fredholm operator as proved in [3], Appendix.

However, if we replace $U_c(\mathcal{H})$ by $U(\mathcal{H})$, $\mathcal{F}\Lambda^0(\mathcal{H})$ by $L$, $U_c(\mathcal{H} \otimes \mathbb{C})$ by $U(\mathcal{H} \otimes \mathbb{C})$, and $\mathcal{F}\Lambda^0_{\text{got}}(\mathcal{H} \otimes \mathbb{C})$ by $L^C$, we have the same results as in [2], §§1.1-1.2:

The group $U(\mathcal{H})$ (resp. $U(\mathcal{H} \otimes \mathbb{C})$) acts transitively on $L$ (resp. on $L^C$).

So let

$$\rho : U(\mathcal{H}) \rightarrow L$$

$$U \mapsto U(\lambda^0)$$

and

$$\rho^C : U(\mathcal{H} \otimes \mathbb{C}) \rightarrow L^C$$

$$g \mapsto g(\lambda^0 \otimes \mathbb{C})$$
CRISS-CROSS REDUCTION OF THE MASLOV INDEX

denote the mappings defined by these actions. We obtain a commutative diagram

\[
\begin{array}{ccc}
U(\mathcal{H}) & \stackrel{\tilde{\tau}}{\longrightarrow} & U(\mathcal{H} \otimes \mathbb{C}) \\
\downarrow^{\rho} & & \downarrow^{\rho^C} \\
\mathcal{L} & \stackrel{\tau}{\longrightarrow} & \mathcal{L}^C,
\end{array}
\]

(A.3)

where \( \tilde{\tau} \) denotes the complexification \( U \hookrightarrow U \otimes \text{Id} \).

Correspondingly to [2], Proposition 1.3 we have

**Proposition A.1.** The mapping

\[ \rho^C \circ \Phi : U(\mathcal{H}) \longrightarrow \mathcal{L}^C \]

is a homeomorphism, where \( \Phi \) is defined in the same way as in [2], p. 7.

Now we must add the following lemma to [2], following Definition 1.4:

**Lemma A.2.** Let \( \lambda \in \mathcal{L} \) and \( \lambda = U(\lambda_0^\perp) \). Then we have that

\[ \lambda \in \mathcal{FL}_{\lambda_0}(\mathcal{H}) \iff U \overline{U}^{-1} + \text{Id} \text{ is a Fredholm operator.} \]

**Proof.** By identifying \( \mathcal{H} \cong \lambda_0 \otimes \mathbb{C} \cong \lambda_0 \oplus \sqrt{-1} \lambda_0 \), we represent \( U \) as

\[ U = X + \sqrt{-1} Y \]

with \( X, Y : \lambda_0 \to \lambda_0 \). Since

\[ U \overline{U}^{-1} + \overline{U} \overline{U}^{-1} = (U + \overline{U}) \overline{U}^{-1} = 2X \overline{U}^{-1}, \]

we have that \( \overline{U} \overline{U}^{-1} + \text{Id} \) is a Fredholm operator, if and only if \( X \) is a Fredholm operator.

First we assume \( \lambda \in \mathcal{FL}_{\lambda_0}(\mathcal{H}) \). Then we know already by [2], Equation (1.1) that

\[ \ker X = \lambda \cap \lambda_0, \]

(A.4)

and we have also

\[ \lambda + \lambda_0 = \{-Y(x) + \sqrt{-1} X(x) + y \mid x, y \in \lambda_0\}. \]

(A.5)

So

\[ \mathcal{H}/(\lambda + \lambda_0) \cong \lambda_0/\text{range}(X), \]

(A.6)

hence \( \text{range}(X) \) is closed and of finite codimension in \( \lambda_0 \).

Conversely, if \( X \) is a Fredholm operator, then also by (A.4), (A.5), and (A.6) we have \( \lambda \in \mathcal{FL}_{\lambda_0}(\mathcal{H}) \). \( \square \)
References


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304a/95 STATISTIKNOTER Simple binomialfordelingsmodeller
af: Jørgen Larsen

304b/95 STATISTIKNOTER Simple normalfordelingsmodeller
af: Jørgen Larsen

304c/95 STATISTIKNOTER Simple Poissonfordelingsmodeller
af: Jørgen Larsen

304d/95 STATISTIKNOTER Simple multinomialfordelingsmodeller
af: Jørgen Larsen

304e/95 STATISTIKNOTER Mindre matematisk-statistisk opslagav
indeholdende bl.a. ordforklaringer, resuméer og
tabeller
af: Jørgen Larsen

306/95 Goals of mathematics teaching Preprint of a chapter for the forthcoming International Handbook of Mathematics Education (Alan J. Bishop, ed) By: Mogens Niss

307/95 Habit Formation and the Thirdness of Signs Presented at the semiotic symposium The Emergence of Codes and Intensions as a Basis of Sign Processes By: Peder Voetmann Christiansen

308/95 Metaforer i Fysikken af: Marianne Wilcken Bjerregaard, Frederik Voetmann Christiansen, Jørn Skov Hansen, Klaus Dahl Jensen Ole Schmidt Vejledere: Peder Voetmann Christiansen og Petir Viscor

309/95 Tiden og Tanken En undersøgelse af begrebsverdenen Matematik udført ved hjælp af en analogi med tid af: Anita Stark og Randi Petersen Vejleder: Bernhelm Booss-Bavnbek

310/96 Kursusmateriale til "Lineære strukturer fra algebra og analyse" (E1) af: Mogens Brun Heefelt

311/96 2nd Annual Report from the project LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM by: Hélène Connor-Lajambe, Bernd Kuemmel, Stefan Krüger Nielsen, Bent Sørensen

312/96 Grassmannian and Chiral Anomaly by: B. Booms-Bavnbek, K.P. Wojciechowski

313/96 THE IRREDUCIBILITY OF CHANGE AND THE OPENNESS OF THE FUTURE The Logical Function of Idealism in Peirce’s Philosophy of Nature By: Helmut Pape, University of Hannover

314/96 Feedback Regulation of Mammalian Cardiovascular System By: Johnny T. Ottesen

315/96 "Rejsen til tidens indre" - Udarbejdelse af et manuskript til en fjermdansudendelse + manuskript af: Gunhild Hune og Karina Goyle Vejledere: Peder Voetmann Christiansen og Bruno Ingemann

316/96 Plasmascillation i natriumlynger Specialrapport af: Peter Neibom, Mikko Østergård Vejledere: Jeppe Dyre & Jørn Borggreen

317/96 Poincaré og symplektiske algoritmer af: Ulla Rasmussen Vejleder: Anders Madsen

318/96 Modelling the Respiratory System by: Tine Ouldager Christiansen, Claus Drøby Supervisors: Viggo Andreasen, Michael Danielsen

319/96 Externality Estimation of Greenhouse Warming Impacts by: Bent Sørensen

320/96 Grassmannian and Boundary Contribution to the Determinant by: K.P. Wojciechowski et al.

321/96 Modelkompetencer - udvikling og afprøvning af et begrebsapparat Specialrapport af: Nina Skov Hansen, Christine Iversen, Kristin Troels-Smith Vejleder: Morten Blohmøj

322/96 OPGAVESAMLING Brede-Kursus i Fysik 1976 - 1996

323/96 Structure and Dynamics of Symmetric Diblock Copolymers PhD Thesis by: Christine Maria Papadakis

324/96 Non-Linearity of Baroreceptor Nerves by: Johnny T. Ottesen

325/96 Retorik eller realitet? Avndelser af matematik i det danske Gymnasiums matematikundervisning i perioden 1903 - 88 Specialrapport af Helle Pilemann Vejleder: Mogens Niss

326/96 Bevissteori: Eksemplificeret ved Gentzenes bevis for konsistens af teorien om de naturlige tal af: Gitte Andersen, Lise Mariane Jeppesen, Klaus Frovin Jørgensen, Ivar Peter Zeck Vejledere: Bernhelm Booss-Bavnbek og Stig Andur Pedersen

327/96 NON-LINEAR MODELLING OF INTEGRATED ENERGY SUPPLY AND DEMAND MATCHING SYSTEMS by: Bent Sørensen

328/96 Calculating Fuel Transport Emissions by: Bernd Kuemmel
329/96 The dynamics of cocirculating influenza strains conferring partial cross-immunity and
A model of influenza A drift evolution
by: Viggo Andreasen, Juan Lin and Simon Levin

330/96 LONG-TERM INTEGRATION OF PHOTOVOLTAICS INTO THE GLOBAL ENERGY SYSTEM
by: Bent Sørensen

331/96 Viskæse fingre
Special rapport af:
Vibeke Orlie and Christina Specht
Vejledere: Jacob M. Jacobsen og Jesper Larsen

332/97 ANOMAL SWELLING OF LIPIDE DOPPELTLAG
Special rapport af:
Stine Sofie Karrmann
Vejleder: Dorthe Posselt

333/97 Biodiversity Matters
an extension of methods found in the literature on monetization of biodiversity
by: Bernd Kuenmmel

334/97 LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM
by: Bernd Kuenmmel and Bent Sørensen

335/97 Dynamics of Amorphous Solids and Viscous Liquids
by: Jeppe C. Dyre

336/97 PROBLEM-ORIENTATED GROUP PROJECT WORK AT ROSKILDE UNIVERSITY
by: Kathrine Legge

337/97 Verdensbankens globale befolkningsprognose
- et projekt om matematisk modellering
af: Jørn Chr. Bendtsen, Kurt Jensen, Per Pauli Petersen
Vejleder: Jørgen Larsen

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Første modul fysikprojekt
af: Søren Dam, Røben Danielsen, Martin Niss, Røben Friis Pedersen, Frederik Resen Steenstrup
Vejleder: Tage Christensen

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by: Wolfgang Coy

340/97 Prime ends revisited - a geometric point of view -
by: Carsten Lunde Petersen

341/97 Two chapters on the teaching, learning and assessment of geometry
by: Mogens Niss

342/97 LONG-TERM SCENARIOS FOR GLOBAL ENERGY DEMAND AND SUPPLY
A global clean fossil scenario discussion paper
prepared by Bernd Kuenmmel
Project leader: Bent Sørensen

343/97 IMPORT/EXPORT-POLITIK SOM REDSKAB TIL OPTIMERET UDNYTTELSE AF EL PRODUCERET PÅ VE-ANLÆG
af: Peter Meibom, Torben Svendsen, Bent Sørensen

344/97 Puzzlen og Siegel diskas
by Carsten Lunde Petersen

345/98 Modeling the Arterial System with Reference to an Anesthesia Simulator
Ph.D. Thesis
by: Mette Sofie Olufsen

346/98 Kløngedannelses i en hulkatode-forstøvningsproces
af: Sebastian Horst
Vejledere: Jørn Borggren, NBI, Niels Boye Olsen

347/98 Verificering af Matematiske Modeller
- en analyse af Danmarks Euleriske Model
af: Jonas Blomqvist, Tom Pedersen, Karen Timmermann, Lisbeth Øhlenschlager
Vejleder: Bernhelm Boose-Bovnbeek

348/98 Case study of the environmental permission procedure and the environmental impact assessment for power plants in Denmark
by: Stefan Krüger Nielsen
Project leader: Bent Sørensen

349/98 Tre rapporter fra FACMAT - et projekt om tal og faglig matematik i arbejdsmarkedskundskabelserne
af: Lena Lindenskov og Tine Wedage

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351/98 Aspects of the Nature and State of Research in Mathematics Education
by: Mogens Niss
352/98 The Herman-Swiatec Theorem with applications
by: Carsten Lunde Petersen

353/98 Problemløsning og modellering i en almindeligt matematikundervisning
Specialrapport af: Per Gregersen og Tomas Højgaard Jensen
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354/98 A GLOBAL RENEWABLE ENERGY SCENARIO
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355/98 Convergence of rational rays in parameter spaces
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Modelprojekt af: Thomas Frommelt, Hans Ravnhøj Larsen og Arnold Skimminge
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357/98 Cayleya Problem
En historisk analyse af arbejdet med Cayley problem fra 1870 til 1918
Et matematisk videnskabsfagsprojekt af: Rikke Degn, Bo Jakobsen, Bjørke X.W. Hansen, Jesper S. Hansen, Jesper Ulden, Peter C. Wulff
Vejleder: Jesper Larsen

358/98 Modeling of Feedback Mechanisms which Control the Heart Function in a View to an Implementation in Cardiovascular Models
Ph. D. Thesis by: Michael Danielsen

359/99 Long-Term Scenarios for Global Energy Demand and Supply Four Global Greenhouse Mitigation Scenarios
by: Bent Sørensen

360/99 SYMMETRI I FYSIK
En Meta-projekt rapport af: Martin Niss.
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361/99 Symplectic Functional Analysis and Spectral Invariants
by: Bernhelm Booss-Bavnbek, Kenro Furutani

362/99 Er matematik en naturvidenskab? – en udsendelse af diskussionen
En videnskabsfagsprojekt-rapport af Martin Niss
Vejleder: Mogens Nørgaard Olesen