Global Dimension, Tower of Algebras, and Jones Index of Split Separable Subalgebras with Unitality Condition

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In Memory of Oscar Goldman, Mathematician and Family Friend
Abstract

This paper addresses a question of M. Pimsner and S. Popa about what properties are shared by a subalgebra and an algebra given a Hilbert module structure with orthonormal basis [26] in the context of von Neumann algebra type $II_1$-factors. A rather complete answer to this question occurs in an algebraic setting that brings together the von Neumann theory with Galois theory and group representations. The setting uses ideas of Higman and Jans from the fifties about relative separability and split extension of finite group algebras over characteristic p. Following R. Pierce we drop the ground field condition and most finiteness conditions, but not a condition of finite index, to obtain viable notions of relative separability and split extension, to which we add a condition we call unitality. The unitality condition has the decided advantage of excluding little (the Higman group algebra and the Auslander-Goldman theory of Galois extensions for commutative rings satisfy unitality) but making much of the Jones theory possible. With unitality we show that global dimension of subalgebra $S$ and algebra $A$ in a split separable extension are the same, a generalization of Serre’s extension theorem. $A$ and $S$ also obtain a basic construction, a Jones index and tower of algebras, which admit representations of the braid groups. Finally, a metatheorem is formulated to show homological properties are shared by subalgebra $S$ and algebra $A$, by Morita equivalent rings and a generalization of these we call split separably equivalent rings.

I wish to extend warm thanks to teachers, family, and colleagues Daniel Kastler and Peter Seibt at C.P.T., C.N.R.S.-Luminy, and Jesus Gonzalo of Universidad Autonoma de Madrid. My current address: Department of Mathematics, University of California, Berkeley, CA, 94720 U.S.A.
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1 Introduction

Let $M$ and $N$ be finite factors, i.e., the set of values of the trace on projections are finite, and the center is trivial, with $N$ an subfactor of $M$ where $1_N = 1_M$. Let $E : M \rightarrow N$ denote the (unique trace-preserving) conditional expectation of $M$ onto $N$, i.e., $E$ is an $N$-$N$ bimodule morphism of the obvious bimodule structures on $M$ and $N$, and $E(n) = n \forall n \in N$.

The basic construction builds a finite factor $M_1$ containing $M$ as a subfactor with properties among which we mention:

1. $M_1$ is singly generated as an $M$-$M$ bimodule by a projection $e_1$.
2. $e_1 me_1 = E(m)e_1 = e_1 E(m)$.
3. Given $m \in M$, $me_1 = 0$ or $e_1 m = 0 \Rightarrow m = 0$.
4. The unique trace-preserving conditional expectation $E_1 : M_1 \rightarrow M$ is given by $E_1(e_1) = \tau 1$ for some positive number $\tau$.

The Jones index $[M:N]$ of $N$ in $M$ is $\tau^{-1}$, a real number between $1$ and $\infty$ with interesting restrictions on its value [12]. Iterating the basic construction leads to a tower of algebras $N \subset M \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots$ and a family of projections $\{e_i\}_{i=1}^\infty$ in the direct limit $M_{\infty}$ (also a finite factor). These projections satisfy relations that permit representations of the infinite braid group and certain infinite Hecke algebras:

(i) $e_i e_{i+1} e_i = \tau e_i$,
(ii) $e_i e_{i+1} e_{i+1} = \tau e_{i+1}$
(iii) $e_i e_j = e_j e_i \quad i - j \geq 2$.

The trace of $M_{\infty}$ plays a critical role in defining the Jones polynomial invariant of knot theory [13].

The next theorem is crucial to this paper.

Theorem 1.1 (Pimsner,Popa [26]) If $[M : N] < \infty$ with $n$ the integer part of $[M:N]$, then there exists a family $\{m_j\}_{j=1}^{n+1}$ of elements in $M$ satisfying the properties:

(a) $E(m_j^* m_k) = 0$, $j \neq k$,
(b) $E(m_j^* m_j) = 1$, $1 \leq j \leq n$;
(c) $E(m_{n+1}^* m_{n+1})$ is a projection in $N$ of trace $[M:N] - n$.
(d) $\sum_{j=1}^{n+1} m_j e_1 m_j^* = 1$;
(e) $\sum_{j=1}^{n+1} m_j m_j^* = [M : N]$.

It follows from the theorem that $M_N$ is a finitely generated projective right (or left, see section 4) $N$-module with dual basis $\{m_j\}_{j=1}^{n+1}$ in $M$ and $\{E(m_j^* -)\}_{j=1}^{n+1}$ in $Hom_N(M, N)$. After giving several applications of their theorem, Pimsner and Popa propose that $N$ and $M$ share many properties for an unspecified reason related to their theorem [26, p. 67]. In this paper we abstract algebraic properties from their theorem that ensures that numerous algebraic properties are indeed shared by subalgebra $N$ and algebra $M$, though $N$ and $M$ are neither isomorphic nor Morita equivalent.
We place three conditions on a (not necessarily involutive but unital) subalgebra \( S \) of \( A \) over a commutative ring \( k \) that ensures that properties of an algebra definable in terms of module categories (with certain mild restrictions) are shared by both \( S \) and \( A \). In addition, from \( S \) and \( A \) we obtain the basic construction, the Jones index, tower of algebras with Morita equivalence of every other algebra in the tower, and countably many idempotents satisfying braid relations. The conditions we impose on the subalgebra \( S \) of \( A \), sometimes referred to in the reverse as the extension \( A \) of \( S \), are the following three with labels for purposes of reference:

(I) **Relative separability.** \( A \) is a separable extension of \( S \), i.e., \( A \) has relative cohomological dimension zero over \( S \) \([10]\), or equivalently, there exists a (separating) element \( e \) in \( A \otimes_S A \) satisfying i) \( \mu_S(e) = 1 \) where \( \mu_S : A \otimes_S A \to A \) is the multiplication map; ii) \( ae = ea \) \( \forall a \in A \) \([16]\).

(II) **Split extension.** \( S \) is a direct summand of \( A \) as \( S \)-\( S \) bimodules, i.e., a conditional expectation \( E : A \to S \) exists.

(III) **Unitality.** There exists a conditional expectation \( E : A \to S \) and separating element \( e = \tau \sum_{i=1}^{n} x_i \otimes y_i \) with \( \tau \in k \) such that

\[
\sum_{i=1}^{n} E(x_i)y_i = 1 = \sum_{i=1}^{n} x_i E(y_i)
\]

If the subalgebra \( S \) of \( A \) satisfies (I), (II), and (III) we refer to \( S \) as a split separable subalgebra of \( A \), or \( A \) as a split separable extension of \( S \). What condition (I) has to do with the finite factors is given by joint work with D. Kastler:

**Proposition 1.1** \([18]\) \( M \) is a separable extension of \( N \) with separating element

\[
\frac{1}{[M : N]} \sum_{j=1}^{n+1} m_j \otimes_S m_j^*.
\]

Together with the unique trace-preserving conditional expectation \( E : M \to N \), it is an exercise in applying (d) and (2) above to show that property (III) is satisfied. The author and Kastler have also shown that type III subfactors of finite Kosiaki index are relatively separable as well. Numerous examples of split separable extensions exist in algebra, most notably subgroups of finite index and the group algebra extensions they generate (or twisted, smashed versions thereof).

A convenience to our paper is the language of change of rings, which we review. For a unital ring homomorphism \( R \to S \), we say an \( S \)-module restricts to the \( R \)-module obtained by using the arrow, while an \( R \)-module \( M \) induces to the \( S \)-module \( S \otimes_R M \). The arrow \( R \to S \) is referred to as a projective, flat or finitely generated change of rings in case \( S \) is projective, flat, or finitely generated, respectively, in the natural \( R \)-module structure induced from the arrow. A property of modules is an assignment to each ring \( R \), of a subclass
Φₚ of all R-modules, R-Mod. A property is said to induce or restrict in case every module in Φₚ induces to one in Φₛ, or every module in Φₛ restricts to one in Φₚ. Note that projectivity or flatness always induce but only restrict under projective or flat change of rings.

Now it is a tautology that a subalgebra S shares property X with A means

- S has property X if and only if A has property X.

Call a property X of modules, Φₚ ⊆ R - Mod, nice if it induces and restricts under a finitely generated projective change of rings, and direct summands of members of Φₚ are again members for each ring R. For example, flatness and projectivity are nice properties. Then call a property Y of rings homological if R has property Y is equivalent to two nice properties of modules coinciding, Φₚ = Φₛ. A simple example of a homological property is von Neumann regularity for rings, which is definable as rings all of whose modules are flat.

For some homological properties such as (weak) global dimension 0, the forward implication in • is given by relative separability alone. The relevant module properties of relative separability is the following:

**Proposition 1.2** A is a separable extension of S iff for every k-algebra C and C-A bimodule N, N is a direct summand of N ⊗ₚ A as C-A bimodules.

As we have seen, projectives and flats do not restrict in general, so it turns out that few properties pass in the reverse implication of • where A is a split extension of S. However, there is a dual to the last proposition that is also useful.

**Proposition 1.3** A is a split extension of S iff for every k-algebra B and B-S bimodule N, N is a direct summand of N ⊗ₚ A as B-S bimodules.

A limited application of split and ¹ separable extensions using restricted and induced modules was given in the elegant

_Higman's Theorem_ [9]. Let F be a field of characteristic p and G a finite group. Then the group algebra F[G] has finitely many indecomposable representations up to isomorphism (has f.r.t.) iff each sylow p-subgroup is cyclic.

This theorem may be proven conceptually by first noting that F[G] is a separable extension (though not F-separable!) of F[H] for any of its sylow p-subgroups H, secondly that F[G] is a split extension of F[H], and thirdly proving by means of propositions 1.2 and 1.3

_Jans's Theorem_ [11]. Let A and S be Artinian algebras, S a subalgebra of A such that A is finitely generated over S as a right module. If A is a split extension of S, then A has f.r.t. implies S has f.r.t. If A is a separable extension of S, then S has f.r.t. implies A has f.r.t.

Now we find adding the unitarity condition (III) to split and separable extensions a bonanza for several reasons. First, it is unrestrictive: consider the many

¹notice our use of "and" to distinguish the absence of property (III): "we bend the grammar to make the mathematics come out right" [19].
examples of split and separable extensions, including Higman's group algebras, that can be shown to satisfy the unitality condition (cf. section 3). Second, in the notation of (III), \( \sum_{i=1}^{n} x_i \otimes y_i \) is the identity with respect to an algebra structure \(^2\) on \( A \otimes_S A \) given by
\[
(a_1 \otimes_S a_2)(a_3 \otimes_S a_4) = a_1 E(a_2 a_3) \otimes_S a_4 \quad (a_i \in A)
\]
We show that \( A \otimes_S A \) is a split separable extension of \( A \), \( A \otimes_S A \) is Morita equivalent to \( S \), whence one implication alone in \( \bullet \) may be proven in showing \( A \) and \( S \) share a property \( X \), since Morita equivalent rings have equivalent categories of modules. Third, we show that \( A \) is a finite projective over \( S \) from either side, as in the Pimsner-Popa theorem: then projectives and flats restrict and homological properties properties pass down to the subalgebra while resolutions of flats and projectives induce up. As a consequence, statement \( \bullet \) above becomes a metathorem for homological properties \( X \) and split separable extensions \( A \) of \( S \). Our main theorem is valid for weak, left, or right global dimension \( D(\cdot) \) of rings:

Theorem 1.2 Suppose \( A \) is a split separable extension of \( S \). Then \( D(A) = D(S) \).

In terms of noncommutative topology, theorem 1.2 may be viewed as an algebraic analogue of the fact that the dimension of a space is the same as any of its quotients by a free action of a finite group of homeomorphisms (cf. example 3.1.4). It is also a generalization of the Serre extension theorem (with a certain restriction in the characteristic \( p \) case cf. section 5, [24], [29]) and scattered results in commutative ring theory (an extension of a Dedekind ring is Dedekind, etc.). Other properties that are necessarily shared by a subalgebra and algebra include that of being a left Noetherian ring, left perfect ring, left coherent ring, quasi-Frobenius ring, or a polynomial identity ring.

In spite of their homological similarities, \( A \) and \( S \) are not Morita equivalent: e.g., the real quaternions are a split separable extension of the reals but not Morita equivalent to them. We clarify the relation of Morita equivalence with split separability by studying a generalization of both we call split separable equivalence of rings. This equivalence is obtained by first weakening the bimodule isomorphism condition of Morita equivalence to only requiring split surjectivity: second, fully symmetrizing the unitality condition.

Our paper proceeds with showing that a split separable subalgebra \( S \) of \( A \) possesses a Jones tower of algebras with index. Indeed, a generalization of the Jones index of subfactors is given by

Definition 1.1 With \( e \) and \( E \) satisfying property (III) of split separable extensions and \( \tau \) invertible in the ground ring \( k \), define the index of \( S \) in \( A \), \([A:S] = \tau^{-1}\).
Fixing the conditional expectation (there is usually a canonical one that one wants to retain), the index \([A:S]\) is independent of the choice of separating element \(e\) so long as (III) is satisfied. Also, \(A\) inside the algebra \(A \otimes_S A\) has the same index. We label \(A \otimes_S A\) the basic construction since it is isomorphic to \(M_1\) where \(A\) and \(S\) are the finite factors \(M\) and \(N\), respectively (cf. [15]). The expectation map \(E_1 : A \otimes_S A \to A\) is then \(\tau\mu_S\), where \(\mu_S\) is the multiplication map, and the idempotent \(e_1 = 1 \otimes 1\) satisfies properties (1)-(4) above (where \(A\) replace \(M\), \(A \otimes_S A\) replaces \(M_1\)). The basic construction may be iterated to obtain a tower of algebras over \(A\) and \(S\) with countable family of idempotents \(\{e_i\}_{i=1}^\infty\) satisfying the relations (i)-(iii) above. We conclude the paper with a condition on the ground ring \(k\) that permits an explicit representation of the braid group on \(n\) letters in the \(n\)'th algebra of the tower.

The paper is organized as follows:

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Note. I learned from Susan Montgomery after writing this preprint of Y. Watatani, Index of \(C^*-\)algebras, Memoirs A.M.S. 85, no. 492 (1990), in which there is some overlap. Late versions of this paper should reflect improvements gleaned from a reading of Watatani.

2 Relative Separability

Throughout this section, let \(k\) be a commutative ring with unit, \(A\) a \(k\)-algebra with subalgebra \(S\) with \(1_A \in S\).

Definition 2.1 A is said to be a separable extension of \(S\) iff there exists an \(e \in A \otimes_S A\) (which we call a separating element) such that

(i) \(\mu_S(e) = 1\) where \(\mu_S : A \otimes_S A \to A\) is the multiplication map, \(a \otimes b \mapsto ab\).

(ii) \(ae = ea\) \(\forall a \in A\).
Example 2.1. This example is known in representation theory of finite groups under a different guise [7]. Suppose \( H \) is subgroup of finite index \([G:H] = n\) of a possibly infinite group \( G \). If \( n \) is invertible in \( k \), then the group algebra \( k[G] \) is a separable extension of \( k[H] \). If \( \{g_i\}_{i=1}^n \) is a cross-section of the left cosets of \( H \) in \( G \), then

\[
c = \frac{1}{[G:H]} \sum_{i=1}^n g_i \otimes_{k[H]} g_i^{-1}
\]

is a separating element. \( k \) may be replaced by any \( k \)-algebra \( A \) with \( G \) embedded in its automorphism group, then twisted or quantum group versions of this example are similarly shown to be separable extensions.

Proposition 2.1 The following conditions on \( A \supseteq S \) are equivalent:

1. \( A \) is a separable extension of \( S \);
2. \( A \) has relative Hochschild cohomological dimension 0 over \( S \);
3. \( A \) is an \( S^e \)-relative projective \( A^e \)-module (where \( B^e \) denotes \( B \otimes_k B^{op} \) for general \( k \)-algebras \( B \));
4. The universal derivation \( d: A \rightarrow A \otimes_S A \) is inner;
5. the module condition in Proposition 1.2
6. Every \( S^e \)-split epi is \( A^e \)-split.

Proof. The equivalence of conditions 1),2),3),4), and 5) may be found in [16] or [2] (terminology from [10] and [20]). We next show the equivalence of 1) with 6). First assume 6). The multiplication map \( \mu_S \) is an epimorphism of \( A^e \)-modules, \( S^e \)-split by the map sending \( a \mapsto a \otimes_S 1 \). Hence there exists an \( A^e \)-splitting \( \eta: A \rightarrow A \otimes_S A \), and \( e = \eta(1) \) is checked to be a separating element. The equivalence of 1) with 5) is Proposition 2.2 below.

Two applications of the next lemma makes an \( S^e \)-split epi of \( A^e \)-modules into a split epi to show that 1) implies 6). \( \square \)

Lemma 2.1 (Trace Lemma) If \( A \) is a separable extension of \( S \), and \( C \) is an arbitrary unital \( k \)-algebra, then a \( C \)-\( S \) split epi of \( A \)-\( A \) bimodules is \( C \)-\( A \) split. Also, an \( S \)-\( C \) split epi of \( A \)-\( A \) bimodules is \( A \)-\( C \) split. Hence, \( A \) has relative global dimension zero over \( S \): short exact sequences of \( A \)-modules that split over \( S \) can be made to split over \( A \).

Proof. \( ^3 \) Let \( e = \sum_{i=1}^n z_i \otimes_S y_i \) be a separating element, \( \sigma: N \rightarrow M \) an epi of \( C \)-\( A \) bimodules with splitting \( f \in \text{Hom}_{C-A}(M,N) \). We now apply a trace operator to alter \( f \) to a \( C \)-\( A \) module morphism \( \gamma \) satisfying \( \sigma \gamma = 1 \).

\( ^3 \) The idea of proof is a generalization of the trace argument proving Maschke's theorem on semisimplicity of finite group algebras.
The trace operator $\text{Tr}_e(-): \text{Hom}_{C-S}(M, N) \otimes_k A \otimes_S A \rightarrow \text{Hom}_{C-k}(M, N)$ is given by

$$\text{Tr}_e(g)(m) = \sum_{i=1}^N g(mz_i)w_i \quad \text{where} \quad z = \sum_{i=1}^N z_i \otimes w_i \quad \text{and} \quad g \in \text{Hom}_{C-S}(M, N).$$

Clearly, $\text{Tr}_e(g)$ is $C$-linear. That $\gamma = \text{Tr}_e(f)$ is $C$-$A$ linear follows from property ii) of separating elements:

$$\text{Tr}_e(f)(ma) = \text{Tr}_e(f)(m) = \text{Tr}_e(f)(m) = \text{Tr}_e(f)(m)a.$$

By property i) we have $\sigma \circ \gamma(m) = \sum_{i=1}^n \sigma(g(mz_i)y_i) = m \sum_{i=1}^n z_iy_i = m$, so $\sigma$ is indeed $C$-$A$ split.

The argument reversing sides for $C$ and $S$ carries through in perfect analogy. If $C = k$ we obtain that short exact sequences of $A$-modules split over $S$ can be made split over $A$ through an application of $\text{Tr}_e$ to the section. $\square$

**Remark 2.1.** It is a consequence of the proof that $\text{Tr}_e(g) \in \text{Hom}_{C-A}(M, N)$ for any $g \in \text{Hom}_{C-S}(M, N)$, and we will refer to this and the corresponding fact where right and left are transposed also as the trace lemma.

**Notation.** If an $R$-module $M$ contains a submodule $N$ as a direct summand, we write $N|M$.

**Proposition 2.2 (1.2)** $A$ is a separable extension of $S$ iff for every $k$-algebra $B$ and $A$-$B$ bimodule $N$, $N$ is a direct summand of $A \otimes_S N$ as $A$-$B$ bimodules.

Similarly, every $B$-$A$ bimodule $N$ is a direct summand of $N \otimes_S A$.

**Proof.** For the back implication, take $N = A$, and note that $A \otimes_S A$ is an $S$-relative projective right or left $A$-module by Frobenius reciprocity [10]. Then $A$ is also an $S$-relative projective module, so the $S$-$A$ split map $\mu_S$ (split by $s \mapsto s \otimes_S 1$) has an $A$-$A$ splitting. It follows routinely that a separating element is given as the image of 1 under an $A$-$A$ splitting of $\mu_S$.

For the forward implication, the multiplication map $\mu : A \otimes_S N \rightarrow N$ given by $a \otimes_S n \mapsto n$ is split by the $S$-$B$ bimodule map $n \mapsto 1 \otimes n$. By the trace lemma we obtain an $A$-$B$ splitting of $\mu$. Hence, $N$ is a direct summand of $A \otimes_S A$. The statement for $B$-$A$ bimodules is proven similarly. $\square$

**Remark 2.2** Relative separability has been referred to as "generic relative global dimension zero" by G. Bergman [2]. We have just seen in the trace lemma that relative separability is stronger than the condition, relative global dimension zero: it is so in very favorable ways which we now discuss (though not central to this paper). Among several interesting classes of separable extensions are ring epimorphisms such as a ring inside a localization [2], a ring $R$ with elements $a$ and $b$ such that $ab = 1$ but $ba \neq 1$ over the subring $S$ generated by 1 and $bR_a$ [2], and matrix rings over an arbitrary algebra [10]. We have already
mentioned the group algebra and von Neumann factor examples above. In addition, separable k-algebras are separable extensions of their unit subalgebra k1. The terminology "separable extension of algebras" is probably due to R. Pierce on the grounds that a finite separable extensions of fields $F_1 \subseteq F_2$ is a separable $F_1$-algebra [25].

The class of separable algebras has closure properties even nicer than those of separable k-algebras. Suppose $A_1$ is a separable extension of $S_1$, and $A_2$ is a separable extension of $S_2$. Then separable extension is closed under direct sum, tensor product, and homomorphic image: i.e., $A_1 \oplus A_2$ is a separable extension of $S_1 \oplus S_2$, $A_1 \otimes_k A_2$ is a separable extension of $S_1 \otimes_k S_2$, and if $f$ is an algebra homomorphism with domain $A_1$ then $f(A_1)$ is a separable extension of $f(S_1)$ [17]. In particular, $f$ may be an automorphism of $A_1$ so that relative separability is a conjugacy invariant property of a subalgebra in the sense of [22]. In addition, if $I$ is an ideal in an algebra $A$ such that $A = A_1 \oplus I$, then $A$ is separable extension of $S = S_1 \oplus I$ [17].

In addition, if $S \subseteq T \subseteq A$ is an intermediate algebra with $A$ a separable extension of $T$, and $T$ a separable extension of $S$, then $A$ is a separable extension of $S$; if $A$ is a separable extension of $S$ only, then $A$ is a separable extension of any intermediate algebra $T$ [25]. Then a separable k-algebra $A$ may separably extend an algebra $S$ that is not semisimple: e.g., $A = M_2(k)$ and $S = a$ triangular algebra inside $A$. This shows that the converse to the next proposition is not true.

**Proposition 2.3** Suppose $A$ is a separable extension of $S$. If $S$ is semisimple (von Neumann regular), then $A$ is semisimple (respectively, von Neumann regular).

**Proof.** Recall that for any subalgebra $S$ of $A$, an $A$-module $N$ may be restricted to an $S$-module $S_N$ while an $S$-module $P$ may be induced to an $A$-module $A \otimes_S P$. Inducing always takes flat modules to flat modules and projectives to projectives. Also recall that a ring $R$ is semisimple (von Neumann regular) iff every left $R$-module is projective (resp., flat).

Begin with any $A$-module $M$. Its restriction $S_M$ is projective (flat) since $S$ is semisimple (resp., von Neumann regular). Then the induced module $A \otimes_S M$ is projective (resp., flat). But $M|A \otimes_S M$ as noted, whence inherits projectivity (resp., flatness). Hence, $A$ is semisimple (resp., von Neumann regular). □

**Remark 2.3** A may not be projective over $S$, as the separable Z-algebra $Q$. Extra conditions on separable extensions are needed for this to happen: conditions (II) and (III) of section 1. Then projective or flat resolutions induce up. We will prove for split separable extensions $A \supset S$, that $S$ has global dimension $n$ iff $A$ has global dimension $n$ (weak, right, or left). What happens to injectives under separable extension is the subject of the next proposition.
Proposition 2.4 Let $A$ be a separable extension of $S$. Then every $A$-module that restricts to an injective $S$-module is itself injective. As a consequence, if $A$ is an integral domain with $S$ a Dedekind domain, then $A$ is a Dedekind domain.

Proof. Any $A$-module $M$ has injective envelope $Q$. Since $S$ is injective, it follows that the inclusion $M \rightarrow Q$ is $S$-split, therefore $A$-split by the trace lemma. Therefore, $M \mid Q$, so $M$ is injective.

Let $A$ be a domain with $S$ a Dedekind domain. By a well-known theorem, it will suffice to consider a divisible $A$-module $M$ and show it is injective [28]. But its restriction $s M$ is trivially divisible, therefore injective, so $M$ is injective. □

3 Split Separable Extensions

In this section we fix the following notation and concepts. Let $k$ be a unital commutative ring, $A$ a $k$-algebra containing a subalgebra $S$ where $S \ni 1_A$. The next definition refers to the natural $S$-$S$ bimodule structures on $S$ and $A$ resulting from multiplication.

Definition 3.1 $A$ is a split extension of $S$ iff as $S$-$S$ bimodules, $S$ is a direct summand of $A$.

Proposition 3.1 The following conditions on a subalgebra $S$ of $A$ are equivalent:

1. $A$ is split extension of $S$;

2. There exists a conditional expectation $E : A \rightarrow S$, i.e., $E|_S = 1d_S$ and $E$ is an $S$-$S$ bimodule morphism;

3. For every $k$-algebra $B$ and $S$-$B$ bimodule $N$, $N$ is a direct summand of $A \otimes_S N$. Similarly, $N|_N \otimes_S A$ for every $B$-$S$ bimodule $N$. (Proposition 1.3.)

Proof. (1) iff (2) results from noting that the inclusion $S \rightarrow A$ splits: $E$ is a choice of splitting. (2) implies (3): Note that $\iota : N \rightarrow A \otimes_S N$ defined by $n \mapsto 1 \otimes_S n$ is split as $S$-$S$ bimodule maps by $E \otimes_S Id_N$ under the obvious identification of $S \otimes_S N$ with $N$. (3) implies (1): Take $N = S$, $B = S$, and make the identification of $A \otimes_S S$ with $A$. □

Example 3.1 1. Let $H$ be a subgroup of a group $G$. Then

$$k[G] = k[H] \oplus \sum_{y \in G \setminus H} kx$$

as $k[H]$-bimodules. An arbitrary $k$-algebra $B$ may replace $k$ in this example, since tensoring preserves direct sum, or introduce a twist given by $G$ acting as automorphisms of $B$. 

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2. Any algebra $A$ over a field $F$ is a split extension over $F$. A von Neumann algebra factor is a split extension over any of its subfactors.

3. Any matrix algebra over $k$ is a split extension of $k$. Polynomial rings and power series rings are split extensions over the coefficient ring.

The next proposition gives an inequality of right (left, or weak) global dimension $D(-)$ of $S$ and $A$ which brings into play the projective dimension of $A$ as an $S$-module.

**Proposition 3.2** If $A$ is a split extension of $S$, then $D(S) \leq D(A) + \text{pr. dim}. A_S$.

**Proof.** Let $M$ be a right $S$-module. By proposition 3.1, $M/M \otimes_S A$, whence pr. dim. $M_S \leq \text{pr. dim.} (M \otimes_S A)_S$. By two well-known change of rings spectral sequences in tor and ext functors, we have pr. dim. $(M \otimes_S A)_S \leq \text{pr. dim.} (M \otimes_S A)_A + \text{pr. dim.} A_S$. Then $D(S) = \sup \text{pr. dim.} M_S \leq D(A) + \text{pr. dim}. A_S$. This argument works for left modules, or weak global dimension by replacing pr. dim with flat dimension of modules. □

**Definition 3.2** $A$ is a split separable extension of $S$ iff the following three conditions are met:

(I) $A$ is a separable extension of $S$;

(II) $A$ is a split extension of $S$;

(III) There exists a separating element $e \in A \otimes_S A$ and conditional expectation $E : A \to S$ such that

$$\mu_S(\text{Id} \otimes_S E)e = \tau 1_A = \mu_S(E \otimes_S \text{Id})e$$

for some nonzero scalar $\tau \in k$. We call this the unitality condition.

**Example 3.2** In each example below $k$ is a commutative ring where $\tau$ is invertible.

1. The group algebra example in section 2. Take $E : k[G] \to k[H]$ to be $E(\sum g \in G a_g g) = \sum g \in H a_g g$. Then

$$\mu_S(E \otimes_S 1)e = \frac{1}{[G : H]} \sum_{i=1}^n E(g_i)g_i^{-1} = \frac{1}{[G : H]} = \mu_S(1 \otimes E)e.$$

Twisted tensor products with an algebra or smash products with a Hopf subalgebra with left integral and antipode may also be checked to be split separable extensions.

2. Von Neumann algebra finite factors $N \subseteq M, N$ a subfactor of $M$ of finite Jones index are split separable extensions (cf. section 1 and [18]).

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3. The full matrix extension $M_n(A)$ of any $k$-algebra $A$ is a split separable extension with separating element

$$ e = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E_{ij} \otimes_A E_{ji}, $$

where $E_{ij}$ is the $(i,j)$-matrix unit, and with conditional expectation

$$ E(X) = \frac{1}{n} \sum_{i=1}^{n} X_{ii} \quad \text{where} \quad X = (X_{ij}) \in M_n(A). $$

Then $e$ and $E$ satisfy condition (III) with $\tau = \frac{1}{n^2}$.

4. Quadratic field extensions $Q(\sqrt{a})$ of the rationals where $a \in Z$ are split separable extensions. Take $E$ to be the normalized trace $E(x + y\sqrt{a}) = x$ and $e = \frac{1}{2}(\sqrt{a} \otimes Q \sqrt{a} + a \otimes 1)$. Then $\tau = \frac{1}{2}$. (Then proposition 2.4 shows that the ring of integers of $Q(\sqrt{a})$ with $2a$ inverted is a Dedekind domain.)

Let $A$ be a Galois extension (of commutative rings) of $S$ with finite group $G$, [1]. Then $A$ is a split separable extension of $S$ if $\tau = \frac{1}{|G|} \in S$. One example of such a Galois extension is a ring of $G$-invariant functions within the ring of continuous complex-valued functions on a compact Hausdorff space where $G$ acts by homeomorphisms without fixed points [4].

5. The quaternion algebras $(a, b)$ over a field $F$ of characteristic $\neq 2$ are a split separable extension of $F$. Apply the trace $E$ to the separability idempotent:

$$ e = \frac{1}{4}(1 \otimes 1 + i \otimes ia^{-1} + j \otimes jb^{-1} - k \otimes ka^{-1}b^{-1}) $$

Note that $\tau = \frac{1}{2}$.

More generally, a crossed product algebra $E \ast G$ of a Galois extension field $E$ of $F$ with Galois group $G$ is a split separable extension of $E$ (whence of $F$ by example 3.4 and proposition 7.3). A long calculation checking that the following is a separating element over $E$:

$$ e = \frac{1}{|G|} \sum_{s \in G} u_s \otimes_E u_s^{-1} $$

where $\delta_{S,T}$ is the defining two-cocycle and $(u_s)_{S \in G}$ is the standard basis of the crossed product algebra.

4 The Basic Construction

We continue to suppose $k$ a commutative ring, and suppose $\tau$ invertible in $k$. The next proposition explains our usage of the term unitality for condition (III).
Proposition 4.1 Suppose $A$ is a split separable extension of $S$ with $E$ and $e = \tau \sum_{i=1}^{n} x_i \otimes y_i$ satisfying property (III). Then $A \otimes_S A$ is a unital algebra with multiplication given by

$$(a_0 \otimes_S a_1)(a_2 \otimes_S a_3) = a_0 E(a_1 a_2) \otimes_S a_3$$

with unity element

$$1 = \sum_{i=1}^{n} x_i \otimes y_i.$$  

Proof. The multiplication is associative because

$$a_0 E(a_1 a_2) E(a_3 a_4) \otimes_S a_5 = a_0 E(a_1 a_2 E(a_3 a_4)) \otimes_S a_5$$

by $S$-linearity of $E$ from both sides.

$\tau^{-1} e$ is the identity by an application of the trace lemma for separable extensions and property (III). With $E \in Hom_{S-S}(A, A)$ the expectation above we have

$$\sum_{i=1}^{n} x_i \otimes y_i(a \otimes b) = \sum_{i=1}^{n} x_i E(y_i a) \otimes b = Tr_{\tau^{-1} e} E(1) \otimes b = a(\sum_{i=1}^{n} x_i E(y_i)) \otimes b = a \otimes b.$$  

Checking that $\tau^{-1} e$ is a right identity is entirely analogous: one makes use of $\sum_{i=1}^{n} E(x_i) y_i = 1$.  \[\square\]

Proposition 4.2 Given the hypotheses and notation of the last proposition, $A \otimes_S A$ is a split separable extension of $A$ under the inclusion $i : a \mapsto a 1$ (\forall a \in A).

Proof. Fix the notation $A_1 = A \otimes_S A$ and $E_1 = \tau \mu_S$ by analogy with the basic construction and the conditional expectation in von Neumann theory. Note that $E_1$ is a conditional expectation since

(i) $\tau \mu_S (a \tau^{-1} e) = \mu_S(e) = a$,

(ii) $\mu_S$ is an $A - A$ bimodule homomorphism.

Now $A_1 \otimes_A A_1 \cong A \otimes_S A \otimes_S A$ as $A_1 - A_1$ bimodules by the map

$$\Upsilon : a_1 \otimes_S a_2 \otimes_A a_3 \otimes_S a_4 \mapsto a_1 \otimes_S a_2 a_3 \otimes S a_4$$

where the bimodule structure on $A \otimes_S A \otimes_S A$ is given by

1) $(a_0 \otimes a_1)(a_2 \otimes a_3 \otimes a_4) = a_0 E(a_1 a_2) \otimes a_3 \otimes a_4$,

2) $(a_0 \otimes a_1 \otimes a_2)(a_3 \otimes a_4) = a_0 \otimes a_1 \otimes E(a_2 a_3) a_4$.

Under the identification by $\Upsilon$, the multiplication map $\mu_A$ is given by

$$a_0 \otimes a_1 \otimes a_2 \mapsto a_0 E(a_1) \otimes_S a_2.$$  

We next claim that the element $f = \sum_{i=1}^{n} x_i \otimes 1 \otimes y_i$ is a separating element, which together with $E_1$ satisfies the unitality condition. We have:
1. $\mu_A(f) \equiv \sum_{i=1}^{n} x_i \otimes y_i = 1_A$

2. $(a_0 \otimes_S a_1)f = \sum_{i=1}^{n} a_0 E(a_1 x_i) \otimes_S 1 \otimes_S y_i$

$$= a_0 \otimes 1 \otimes \sum_{i=1}^{n} E(a_1 x_i) y_i = a_0 \otimes 1 \otimes a_1$$

$$= \sum_{i=1}^{n} x_i E(y_i a_0) \otimes 1 \otimes a_1 = f(a_0 \otimes a_1),$$

by the trace lemma and property (III) for $e$ and $E$;

3. $\mu_A(1 \otimes E_1)f = \mu_A(\tau \sum_{i=1}^{n} x_i \otimes 1 \otimes y_i)$

$$= \tau \sum_{i=1}^{n} x_i \otimes y_i = \tau 1 = \mu_A(E_1 \otimes 1)f.$$

Hence, $A_1$ is a split separable extension of $A$. □

**Proposition 4.3** If $A$ is a split separable extension of $S$, then $A$ is a finitely generated projective left or right $S$-module (in the natural module structures from multiplication).

**Proof.** We prove that $A_S$ is a finite projective - the left module case being entirely analogous. Let $E$ be a conditional expectation and $e = \tau \sum_{i=1}^{n} x_i \otimes y_i$ a separating element satisfying the unitality condition. Then we claim that $\phi_i \in Hom_S(A, S)$ given by $\phi_i(x) = E(y_i x), i = 1, \ldots, n$, and $x_i \in A, i = 1, \ldots, n,$ forms a dual basis for $A_S$, which would complete the proof. But the trace lemma and condition (III) give

$$\sum_{i=1}^{n} x_i E(y_i a) = Tr_{r^{-1}e} E(1) = a \sum_{i=1}^{n} x_i E(y_i) = a. \quad □$$

**Proposition 4.4** If $A$ is a split separable extension of $S$, then $A \otimes_S A$ with the unital algebra structure above is Morita equivalent to $S$.

**Proof.** We prove that $A \otimes_S A$ is isomorphic to the algebra $End_S(A)$ of right $S$-module endomorphisms of $A$. But $A_S$ is finite projective by the previous proposition, and is a generator in the category of right $S$-modules, since the trace ideal in $S$ is all of $S$:

$$\{ \sum_{i=1}^{n} \phi_i(x) | \phi_i \in Hom_S(A, S), x \in A \} \supseteq \{ E(x) | x \in A \} = S.$$

so $S$ and $End_S(A)$ are Morita equivalent [6].

Note the idempotent in $A_1$

$$e_1 = 1 \otimes_S 1$$

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and note that each element of $A_1$ is of the form $\sum_{i=1}^{N} a_i e_1 b_i$, where $a_i, b_i \in A$. Moreover, for each $a \in A$ the multiplication is determined by the relations,

$$e_1 a e_1 = e_1 E(a) = E(a)e_1$$

by a little computation.

Now, $E$ is an idempotent in $\text{End}_S(A)$ since $\text{im} E = S$ and $E[1] = 1d_S$. For each $a \in A$ let $\lambda(a)$ denote left multiplication by $a$, certainly an element of $\text{End}_S(A)$. We claim that the linear map $\theta : A_1 \rightarrow \text{End}_S(A)$ defined by

$$\theta(a e_1 b) = \lambda(a)E\lambda(b)$$

is an isomorphism of $k$-algebras.

$\theta$ is a homomorphism: this follows from noting $E\lambda(b)E = \lambda(E(b)E = E\lambda(E(b))$.

$\theta$ is surjective: given $g \in \text{End}_S(A)$ and $a \in A$,

$$g(a) = g(\sum_{i=1}^{n} x_i E(y_i a)) = \sum_{i=1}^{n} g(x_i)E(y_i a)$$

so $\theta(\sum_{i=1}^{n} g(x_i e_1 y_i)) = g$.

$\theta$ is injective: this is a three step proof using techniques of Morita theory. First, the left $S$-module morphism $\eta : A \rightarrow \text{Hom}_S(A, S)$ given by $a \mapsto E(a\cdot \cdot)$ is injective since $E(ab) = 0 \ \forall b \in A$ implies $a = \sum_{i=1}^{n} E(ax_i)y_j = 0$ ($E$ is then called faithful [8]). Second, $A_S$ is projective, whence flat by the last proposition, so $1 \otimes \eta : A \otimes_S A \rightarrow A \otimes_S \text{Hom}_S(A, S)$ is injective. Third, $\theta$ factors as $\psi (1 \otimes \eta)$ where $\psi : A \otimes_S \text{Hom}_S(A, S) \rightarrow \text{End}_S(A)$, defined by $\psi(a \otimes \gamma) = a\gamma(-)$, is injective by the following calculation:

$$\text{Given } \psi(\sum_{i=1}^{N} a_i \otimes \eta_i) = \sum_{i=1}^{N} a_i \eta_i = 0,$$

$$\text{then } \sum_{i=1}^{N} a_i \otimes \eta_i = \sum_{i=1}^{N} a_i \otimes \eta_i \sum_{j=1}^{n} x_j E(y_j \cdot \cdot) =$$

$$\sum_{i,j} a_i \otimes \eta_i (x_j)E(y_j \cdot \cdot) = \sum_{i,j} a_i \eta_i (x_j) \otimes_S E(y_j \cdot \cdot) = 0.$$

Since $\theta$ is factored into injective maps, $\theta$ is injective. $\Box$

Remark 4.1 Note in the last proof that $\lambda$ will always inject $A$ into $\text{End}_S(A)$, but that we need $A$ a split, finitely generated, projective extension of $S$ to ensure $\text{End}_S(A)$ Morita equivalent to $S$.

We call $A_1$ the basic construction for split separable extensions as a natural extension of terminology from von Neumann theory. Recall from section 1 that
if $A$ and $S$ are the finite factors $M$ and $N$, respectively, with $[M:N] < \infty$, then $M_1 \cong M \otimes_N M$ as $C$-algebras.

Let $k$ be a field. It follows from the last proof and the observation that $S \rightarrow A_1$ given by $s \mapsto se_1$ is injective, that $A_1$ is an $E$-extension of the faithful conditional expectation, $E : A \rightarrow S$ [8]. However, the converse is not true: if $A \otimes_S A$ is an $E$-extension with respect to a split extension $A$ of $S$ with faithful conditional expectation $E : A \rightarrow S$, then $A$ may not be a split separable extension. For example, take $k$ to be a field of characteristic $p$, take $G = Z_p \times Z_p$, a product of two cyclic groups of prime order, and take $H$ to be the left factor $Z_p$. Then $k[G]$ has infinitely many non-isomorphic indecomposable representations, while $k[H]$ has only finitely many by Higman's theorem (cf. Introduction). Then $k[G]$ is not a separable extension of $k[H]$ by Jans's theorem (cf. Introduction), although the conditional expectation defined in example 3.2 is faithful and makes $A \otimes_S A$ a unital $E$-extension.

An alternative definition of a split separable extension $A$ of $S$ is in fact the following: $A$ is a split extension of $S$ with conditional expectation $E : A \rightarrow S$ such that $A \otimes_S A$ is a unital algebra with $\mu_S$ a scalar multiple of a conditional expectation.

5 Global Dimension of Algebra and Subalgebra

The next theorem considerably sharpens the results of propositions 2.3, 2.4, and 3.2 for split separable extensions.

**Theorem 5.1** If $A$ is a split separable extension of a subalgebra $S$, then (weak, right, or left) global dimension $D(A) = D(S)$.

**Proof 1.** Since $A$ is a split extension of $S$, it follows from Proposition 3.2 that

$$D(S) \leq D(A) + \text{ proj. dim. } S.$$  

But $A$ is a projective left $S$-module by proposition, so $D(S) \leq D(A)$. Since the basic construction $A_1$ is a split separable extension of $A$, we also have $D(A) \leq D(A_1)$. But $A_1$ is Morita equivalent to $S$, so $D(A_1) = D(S)$. Then

$$D(S) \leq D(A) \leq D(S),$$  

whence $D(S) = D(A)$. □

**Proof 2.** We need only assume that $A$ is a separable, split, projective extension of $S$ for this proof. Suppose $D(S) = n$, and that $M$ is any left $A$-module. Then its restriction $sM$ has projective dimension at most $n$: i.e., there exists a projective resolution of $M$,

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow sM \rightarrow 0.$$
Some of the projectives above might be zero. Now projective resolutions will induce since $A_S$ is projective, whence flat:

$$0 \rightarrow A \otimes_S P_n \rightarrow A \otimes_S P_{n-1} \rightarrow \cdots \rightarrow A \otimes_S P_0 \rightarrow A \otimes_S M \rightarrow 0$$

Hence, pr. dim. $A \otimes_S M \leq n$. Since $A$ is a separable extension of $S$, $M$ is a direct summand of $A \otimes_S M$ by proposition 1.2. Hence, pr. dim. $M \leq D(S)$, and taking a supremum over all $A$-modules we get $D(A) \leq D(S)$.

Let $N$ be an arbitrary left $S$-module. If $D(A) = m$, then pr. dim. $(A \otimes_S N) \leq m$, i.e., there exists a projective resolution of $A \otimes_S N$,

$$0 \rightarrow P_m \rightarrow \cdots \rightarrow P_0 \rightarrow A \otimes_S N \rightarrow 0.$$

Since $A \otimes_S N$ is projective, this projective resolution restricts to the same for $s(A \otimes_S N)$. But $N|_S(A \otimes_S N)$ since $A$ is a split extension of $S$. It follows that pr. dim. $N \leq m = D(A)$. Hence, $D(S) \leq D(A)$. Putting the two inequalities together we conclude $D(A) = D(S)$.

Note that right modules may replace left modules in the proof above, since $A$ is projective over $S$ from both sides. Also, flat modules and resolutions may replace projective modules and resolutions, since $A$ projective over $S$ implies $A$ is flat over $S$. \square

Remark 5.1 This theorem has a non-trivial intersection with the Serre extension theorem for groups (the excluded case: $k$ = field of characteristic $p$ where $p | [G : H]$, c.f., [24], [29], p. 98).

We can make use of knowing $A_1$ Morita equivalent to $S$ in the next statement about “higher commutativity.”

Proposition 5.1 Suppose $k$ is a field and suppose $A$ is a split separable extension of a polynomial identity algebra $S$. Then $A$ is a polynomial identity algebra.

Proof. If $S$ is a polynomial identity algebra, then so is $M_n(S)$ by a theorem of Regev [27]. Trivially, any subalgebra (not necessarily unital) of a polynomial identity algebra is itself satisfying the same polynomial identity. But $A$ is a subalgebra of the basic construct $A_1$, which is Morita equivalent to $S$, whence of the form $gM_n(S)g$ for some integer $n$ and idempotent $g$ in $M_n(S)$. Hence, $A$ satisfies a polynomial identity. \square

The next proposition deals with chain conditions on one-sided ideals.

Proposition 5.2 If $A$ is a split separable extension of $S$, then $S$ is left (respectively, right) Noetherian iff $A$ is left (resp., right) Noetherian.

Proof. The forward implication will use only that $A$ is finitely generated (f.g.) over $S$. Suppose $M$ is an f.g. left $A$-module, and $N$ is any $A$-submodule of $M$. It will suffice to show that $N$ is f.g. Since $sA$ is f.g. by proposition, it follows that the restriction $sM$ is f.g. Since $S$ is left Noetherian, the submodule $sN$ of $sM$
is also f.g. Then $A \otimes_S N$ is f.g. over $A$. But $N$ is the image of $A \otimes_S N$ under an $A$-module map, so $N$ is f.g. Since $A_S$ is f.g. this argument transposes from left to right.

The reverse implication depends only on a f.g. split extension. Let $M$ be a f.g. left $S$-module with $N$ a submodule. Then $A \otimes_S N$ is an $A$-submodule of $A \otimes_S M$, the latter being f.g. so the former is f.g. since $A$ is left Noetherian. Then $s(A \otimes_S N)$ if f.g. But $N|_S(A \otimes_S N)$ since $A$ is a split extension of $S$, so $N$ is f.g. □

Remark 5.2 Now the interested reader may easily prove that $A$ and $S$ are both quasi-Frobenius, left perfect, or left coherent, or both are not. In spite of being so similar homologically, it is clear from the many examples in 3.1 that $A$ and $S$ are not necessarily Morita equivalent. Indeed, a property that Morita equivalent rings share that is not shared by $A$ and $S$ is that of being a simple ring. For example, it is easy to compute that $k \times k$ is a split separable extension of $k$. However, in the next section we clarify the relation of split separable extension and Morita equivalence.

6 A Metatheorem for Split Separably Equivalent Rings

It is clear that behind the results of section 5 is a metatheorem, much like a metatheorem would exist for properties shared by Morita equivalent rings. In this section, we define an equivalence relation among rings that we call split separable equivalence. We then show that $A$ and $S$ are split separably equivalent if $A$ is a split separable extension of $S$, or if $A$ is Morita equivalent to $S$. We then prove a metatheorem that such $A$ and $S$ share homological properties.

Definition 6.1 Rings $A$ and $B$ are split separably equivalent if there exist bimodules $A \otimes_B Q$ and $B \otimes_A P$, with split surjections as $A$-$A$ and $B$-$B$ bimodule morphisms, respectively,

$$\nu : \quad P \otimes_B Q \rightarrow A$$

$$\mu : \quad Q \otimes_A P \rightarrow B$$

and elements of adjunction $\sum_{j=1}^n p_j \otimes q_j$ and $\sum_{i=1}^m q'_i \otimes p'_i \quad (\forall p, q \in Q, q' \in Q)$

$$\sum_{i=1}^m \nu(p \otimes q_i) p'_i = p \quad \sum_{i=1}^m q'_i \nu(p'_i \otimes q) = q$$

$$\sum_{j=1}^n \mu(q \otimes p_j) q'_j = q \quad \sum_{j=1}^n p_j \mu(q_j \otimes p) = p$$
Remark 6.1 The last four conditions above state that the functors $F = P \otimes -$:
B-Mod $\rightarrow$ A-Mod, and $G = Q \otimes -$: A-Mod $\rightarrow$ B-Mod, form adjunctions in either
order. They also entail that $P$ and $Q$ are progenitors as $A$- and $B$-modules.
The first two conditions above say that the counits of these adjunctions are split
epis. This definition can be carried right over to categories: adjunctions and
split epis are closed under composition, so the transitivity implied by the word
“equivalence” does indeed hold (and may be checked directly by hand with the
definition above). Symmetry and reflexivity of the relation among rings and
categories is obvious.

Interestingly enough, any additive category $C$ is split separably equivalent in
this sense to a finite product of itself $C \times \cdots \times C$, since the diagonal functor has
left adjoint the coproduct, right adjoint the product [21, p. 87], which coincide
in the biproduct, and both counits of adjunction are split epis.

Note that a certain amount of the structure in section 4 carries over to
$P \otimes_B Q$ and $Q \otimes_A P$ as well as a definition of index $[A:B]$ and $[B:A]$ as in section
7. We will define this more precisely in a forthcoming note.

Proposition 6.1 If $A$ is a split separable extension of $S$, then $A$ and $S$ are split
separably equivalent.

Proof. Take $P = Q = A$ and $B = S$ in definition 6.1. Note that the mul-
tiplication map $\mu_S : A \otimes_S A \rightarrow A$ is a split surjection of $A$-$A$ bimodules by
proposition 2.2. Given conditional expectation $E : A \rightarrow S$, we get the following
split surjection of $S$-$S$ bimodules, $A \otimes_A A \cong A \xrightarrow{E} S$. Finally, $SA$ and $AS$ are
finite projectives by proposition 4.3, while $AA$ and $AA$ are free of rank 1.
The elements of adjunction in $P \otimes Q$ and $Q \otimes P$ are given by $\sum x_i \otimes y_i$ and
$1 \otimes 1$, respectively, if the former is the separating element up to $\tau$ satisfying
the unitality condition. The two equations involving $\mu_S$ are trivial, the two
involving $E$ are precisely the content of the unitality condition and the trace
lemma, as noted in 4.3, $\sum E(ax_i)y_i = a$ and $\sum x_iE(y_i a) = a$. \qed

Proposition 6.2 If $A$ and $B$ are Morita equivalent rings, then $A$ and $B$ are
split separably equivalent.

Proof. It is well-known that one of several equivalent ways to define Morita
equivalent rings $A$ and $B$ is to stipulate bimodules $A P_B$ and $B Q_A$ that satisfy

$$P \otimes_B Q \cong A$$

$$Q \otimes_A P \cong B$$
as $A$-$A$ and $B$-$B$ bimodules, respectively where the bimodule isomorphisms are
associative (cf. [6]). The elements of adjunction are then the inverse images of
$1_A$ and $1_B$, and associativity yields the four equations of adjunction. \qed

Nice Properties of Modules. We shall informally say that a property of
left modules, such as projectivity or flatness, is a choice of subclass $\Phi_R$ of all
R-modules, \textbf{R-mod}, for each ring R. A property of modules is said to \textit{induce} (under a finite projective change of rings) if given any two rings R and S and a bimodule \( R_P S \), which is finite projective on either side, then \( M \in \Phi_S \Rightarrow P \otimes_S M \in \Phi_R \). A property of modules is \textit{direct sum invariant} if \( M \in \Phi_R, N \mid M \Rightarrow N \in \Phi_R \) for all rings \( R \). A property of modules is \textit{nice} if it is both direct sum invariant and induces under a finite projective change of rings. For example, both flatness and projectivity are nice properties.

\textbf{Homological Properties of Rings.} It is well-known that certain desirable properties of rings are expressible in terms of the coincidence of classes of modules. For example, "all modules are projective (flat)" or "all modules are quotients of projectives" describe important classes of rings [28]. We capture this idea as follows. Define a property of rings to be a certain binary valued function on rings, assigning to each ring a yes or no. Define a \textit{homological} property of rings to be yes-valued precisely on rings \( R \) where two nice properties of modules coincide, \( \Phi_R = \Psi_R \).

\textbf{Metatheorem 6.1} If \( A \) and \( B \) are split separably equivalent rings, then \( A \) and \( B \) share homological properties.

\textbf{Proof.} Let \( \Phi_R \) and \( \Psi_R \) be nice properties of modules, and suppose \( \Phi_A = \Psi_A \). It suffices to prove by symmetry that \( \Phi_B \subseteq \Psi_B \). Suppose \( AP_B \) and \( BQ_A \) satisfy the finite projectivity and split surjectivity conditions of definition 6.1. Given \( M \in \Phi_B, P \otimes_B M \in \Phi_A \) since nice properties induce, whence \( P \otimes_B M \in \Psi_A \) by assumption. Again by inducing \( Q \otimes_A P \otimes_B M \in \Psi_B \). But \( M \mid Q \otimes_A P \otimes_B M \) by one of the split surjectivity conditions. So \( M \in \Psi_B \) by direct sum invariance of nice properties. Hence, \( \Phi_B \subseteq \Psi_B \). \( \square \)

Note that the proof of the metatheorem requires only the weaker notion of split separable equivalence present in an earlier version of my paper where the elements of adjunction are replaced with the assumption that \( P \) and \( Q \) are finite projective as \( A \)- and \( B \)-modules. We next prove that the split surjections carry over to the Tor functors on modules in the following way.

\textbf{Proposition 6.3} Let \( A \) and \( B \) be rings where \( \mu_A : P \otimes_A Q \to B \) and \( \mu_B : Q \otimes_B P \to A \) are split epimorphisms of \( B \)-\( B \) and \( A \)-\( A \) bimodules, respectively, with \( B P_A \) and \( A Q_B \) bimodules finite projective on either side. Then for arbitrary \( A \)-modules \( M_A \) and \( A N \), there is a split epi \( \mu_n : \text{Tor}_n^B(M \otimes_A Q, P \otimes_A N) \to \text{Tor}_n^A(M, N) \) induced from the map \( \text{Id}_M \otimes \mu_B \otimes \text{Id}_N \).

\textbf{Proof.} If \( X \to M \) is a projective resolution of \( M_A \), then \( X \otimes Q \to M \otimes Q \) is a projective resolution of \( M \otimes Q \) since \( A Q \) is flat and \( Q_B \) is projective. If \( \mu_B \) is split by an \( A \)-\( A \) bimodule map \( \sigma \), then \( \sigma(1) = \sum q_i \otimes p_i \) is readily seen to
satisfy $\sum q_i p_i = 1$ and $a \sum q_i \otimes p_i = \sum q_i \otimes p_i a$. Define two maps as follows - they are morphism of complexes by naturality:

$$\begin{align*}
X \otimes_A N &\longrightarrow X \otimes Q \otimes P \otimes N \\
(z \otimes n) &\longmapsto \sum z \otimes q_i \otimes p_i \otimes n \\
X \otimes Q \otimes P \otimes N &\longrightarrow X \otimes_A N \\
(z \otimes q \otimes p \otimes n) &\longmapsto z \mu_B(q \otimes p) \otimes_A n
\end{align*}$$

Now $g \circ f = Id$ since $\mu_B(\sum q_i \otimes p_i) = \mu_B(1) = 1$. Then passing to the homology groups of these two complexes, $g$ induces the split surjection $\mu_n$ as claimed. \qed

Not surprisingly, global dimension \( \leq n \) is a homological property of rings as revealed by a close examination of proof 2 of theorem 5.1. However, the last proposition provides a convenient proof of the following.

**Corollary 6.1** If $A$ and $B$ are split separably equivalent rings, then $D(A) = D(B)$. 

**Proof.** In the notation of proposition 6.3, $\text{Tor}^A_n(M, N) \mid \text{Tor}^B_n(M \otimes_A Q, P \otimes_A N)$, so that $D(B) \geq D(A)$. By the symmetry in our definition of Jones equivalence, we also get $D(A) \geq D(B)$. \qed

## 7 The Jones Index and Tower of Algebras

Note that in each of the five examples given in section 3 $\tau = \mu_S(E \otimes_A 1)e$ is the inverse of the Jones index defined in [12] and more broadly in [8] and [15]. This suggests the next definition and proposition. $k$ remains a commutative ground ring in which $\tau$ is invertible.

**Definition 7.1** Given a split separable extension $A$ of $S$ with conditional expectation $E$ and separating element $e$ satisfying unitality, $\mu_S(E \otimes 1)e = \tau 1_A = \mu_S(1 \otimes E)e$, define the index of $S$ in $A$, $[A:S] = \tau^{-1}$.

Note that the index is an element of $k$, not a positive real unless extra conditions are attached to split separable extensions. The next proposition shows that if the conditional expectation is fixed, as it often is in operator algebras, the index is well-defined for split separable extensions.

**Proposition 7.1** Suppose $A$ is a split separable extension of $S$ with conditional expectation $E$ and two separating elements $e$ and $e'$ both satisfying the unitality condition. Suppose $\mu_S(E \otimes 1)e = \tau$ and $\mu_S(E \otimes 1)e' = \tau'$. Then $\tau = \tau'$. 

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Proof. Multiplying \( e = \tau \sum_{i=1}^{n} z_i \otimes_S y_i \) and \( e' = \tau' \sum_{j=1}^{m} z'_j \otimes_S y'_j \) in \( A \otimes_S A \) gives

\[
e e' = \tau \tau' \sum_{i,j} (z_i \otimes y_i) (z'_j \otimes y'_j) = \tau \tau' \sum_{j=1}^{m} \left( \sum_{i=1}^{n} z_i E(y_i) z'_j \otimes_S y'_j \right)
\]

\[
= \tau \tau' \sum_{j=1}^{m} z'_j \left( \sum_{i=1}^{n} z_i E(y_i) \right) \otimes y'_j = \tau e'
\]

Similarly, one checks that \( ee' = \tau e \). Then \( \tau e' = \tau' e \), so applying the multiplication map \( \mu_S \) we get \( \tau = \tau' \). □

Proposition 7.2 If \( A \) is a split separable extension of \( S \) with basic construction \( A_1 \), then \([A_1 : A] = [A:S] \).

Proof. We refer to the proof and notation in Proposition 4.2 where it was shown that \( A_1 \) is a split separable extension of \( A \). With \( \tau = [A:S]^{-1} \), equation 3 shows that \([A_1:A] = \tau^{-1}\). □

Lemma 7.1 If \( A \) is a split extension of \( S \), then \( 1 \otimes_S a = 0 \) or \( a \otimes_S 1 = 0 \) implies \( a = 0 \).

Proof. The functors \( - \otimes_S A \) and \( A \otimes_S - \) are exact on split exact sequences of \( S \)-modules. Therefore, the split exact sequence \( 0 \rightarrow S \rightarrow A \) gives the exact sequence \( 0 \rightarrow S \otimes_S A \rightarrow A \otimes_S A \). Under the identification \( S \otimes_S A \cong A \), \( a \mapsto 1 \otimes_S a \). Hence, \( 1 \otimes_S a = 0 \) implies \( a = 0 \). Similarly, \( a \otimes_S 1 = 0 \) implies \( a = 0 \). □

Remark 7.1 From the lemma we see that \( S \) is isomorphic to the centralizer algebra of \( e_1 \) in \( A_1 \), via a map \( \Phi : S \rightarrow (A_1)_{e_1} = \{ z \in A_1 : e_1 z = z e_1 = z \} \) given by \( x \mapsto 1 \otimes_S x \).

Proposition 7.3 Suppose \( A \) is a split separable extension of \( B \), and \( B \) is a split separable extension of \( C \). Then \( A \) is a split separable extension of \( C \) with index \([A:C] = [A:B][B:C] \).

Proof. Let \( E_1 : A \rightarrow B \) and \( E_2 : B \rightarrow C \) be the conditional expectations that together with the separating elements \( e_1 = \tau_1 \sum_{i=1}^{n} u_i \otimes_B v_i \) and \( e_2 = \tau_2 \sum_{j=1}^{m} z_j \otimes_C y_j \) satisfy the unitality condition.

It is easy to check that \( E = E_2 \circ E_1 : A \rightarrow C \) is a conditional expectation. We claim that \( e = \tau_1 \tau_2 \sum_{i=1}^{n} u_i z_i \otimes_C y_i v_i \) is a separating element. Trivially, multiplication \( \mu : A \otimes_C A \rightarrow A \) sends \( e \) to 1. We obtain \( ae = ea \) for all \( a \in A \) as follows. If \( M \) is a \( B \)-\( B \) bimodule denote the \( B \)-centralized subgroup of \( M \) by \( M^B = \{ m \in M : bm = mb \ \forall b \in B \} \). In analogy with the trace lemma define an obvious function \( \Psi : A \otimes_B A \otimes (B \otimes_C B) \rightarrow A \otimes_C A \), such that \( ea \) and \( ae \) belong to image under this map of the same point.
Finally, $E$ and $e$ satisfy the unitality condition in a computation that applies the trace lemma:

$$
\mu(1 \otimes S \ E) e = \tau_1 \tau_2 \sum_i \sum_j u_i x_j E_2 \circ E_1(y_j u_i) = \\
\tau_1 \tau_2 \sum_i u_i (\sum_j x_j E_2(y_j E_1(u_i))) = \tau_1 \tau_2 \sum_i u_i E_1(u_i) \sum_j x_j E_2(y_j) = \tau_1 \tau_2
$$

Similarly, $\mu(E \otimes S 1) e = \tau_1 \tau_2$. Hence, $A$ is a split separable extension of C with $[A:C] = [A:B][B:C]$. □

Let $A$ be a split separable extension of $S$. Fix a conditional expectation $E : A \to S$. Suppose $\tau = [A : S]^{-1}$. In the next theorem we iterate the basic construction to obtain a tower of algebras above $A$, i.e. each unital $k$-algebra is a subalgebra via a canonical inclusion of the next algebra, with a countable family of idempotents satisfying braid-like relations. This is an important point since it is demonstrated in [13] that these relations permit representations of the infinite braid group.

**Theorem 7.1** There is a tower of algebras

$$
S \to A \to A_1 \to \cdots \to A_i \to A_{i+1} \to \cdots
$$

where each $A_i$ $(i = 1, 2, \ldots)$ is the basic construction for the split separable extension $A_{i-1}$ of $A_{i-2}$ (where $A_0 = A$ and $A_{-1} = S$) with index $\tau^{-1}$ and conditional expectation $E_{i-1} = \tau \mu_{A_{i-2}}$ (but $E_0 = E$). The family of idempotents $\{e_i\}_{i=1}^\infty$ determined by $e_i = 1_{A_{i-1}} \otimes A_{i-2} 1_{A_{i-1}}$ satisfy the relations:

1. $e_{i+1} e_i e_{i+1} = \tau e_{i+1}$;
2. $e_i e_{i+1} e_i = \tau e_i$;
3. $e_i e_j = e_j e_i \quad i - j \geq 2$.

**Proof.** The properties of the basic construction are given in propositions 4.1 through 4.4 and 7.2. In the proof of 4.4 the idempotent $e_i$ of $A_i$ is defined and shown to be cyclic in that $A_i = A^* e_i$ is a cyclic $A^*$-bimodule, it is shown to satisfy, for each $a \in A$,

$$
(*) \quad e_i a e_i = E_{i-1}(a) e_i.
$$

In lemma 7.1 we noted that $e_i$ is sorting in that $ae_i = 0$ or $e_i a = 0 \Rightarrow a = 0$. In remark 6.1 we have noted that $e_i$ centralizes elements of $S$. Finally, one sees instantly that the conditional expectation $E_i = \tau \mu_{A_{i-2}}$ satisfies $E_i(e_i) = \tau$.

Since $A_i = A_{i-1} \otimes_{A_{i-2}} A_{i-1}$ with multiplication determined by $E_{i-1}$ and unity element 1 by a separating element satisfying the unitarity condition, $A_i$ is an $A_{i-1}$-bimodule and the arrow $A_{i-1} \to A_{i-2}$ is just the map $a \mapsto a1 \quad (\forall a \in A_{i-1})$. It is clear that to complete the proof we have only to check the relations
1.3 above for the cyclic sorting idempotents $e_i$ ($i = 1, 2, \ldots$) satisfying (*) and centralizing elements of $A_{i-2}$.

Now it follows from (*) that $e_{i+1}e_ie_{i+1} = E_i(e_i)e_{i+1} = \tau e_{i+1}$. Hence, we have relation (1). Since $e_{i+1}$ is sorting we have from (1) that

$$e_ie_{i+1}e_ie_{i+1} = \tau e_{i+1} \Rightarrow e_ie_{i+1}e_i = \tau e_i$$

whence (2) follows. Since $e_j \in (A_i)e_i$ if $i - j \geq 2$, then $e_i$ and $e_j$ commute, whence relation (3) follows. \qed

**Corollary 7.1** Same hypotheses and notation as in theorem 7.1. If the ground ring $k$ has an invertible solution $q$ of $q^2\tau = q - 1$, then each braid group on $n$ letters, $B_n$, maps into $A_{n-1}$ under a homomorphism of $k$-algebras $\Phi_n : k[B_n] \rightarrow A_{n-1}$.

**Proof.** It is a classical fact of E. Artin that $B_n$ has a finite presentation,

$$B_n = \{g_1, \ldots, g_{n-1} | g_ig_j = g_jg_i, \ g_{i+1}g_{i+1} = g_ig_{i+1}g_i, \forall i,j \ |i-j| > 1\}$$

In order to define $\Phi_n$ it is sufficient to assign an invertible value $\Phi_n(g_i)$ in $A_{n-1}$ and check the Artin relations. We define

$$\Phi_n(g_i) = qe_i - 1 \quad (i = 1, \ldots, n - 1).$$

$qe_i - 1$ is invertible since $(qe_i - 1)(qe_i + (1 - q)) = 1 - q$ and $1 - q$ is invertible in $k$ since $q$ and $\tau$ are invertibles.

The relation $\Phi_n(g_i)\Phi_n(g_j) = \Phi_n(g_j)\Phi_n(g_i) \ |i-j| > 1$ is clear from relation 3 in theorem 6.1.

Using relation 1 and 2 of theorem 7.1 and the fact that the $e_i$'s are idempotents, we get

$$\Phi_n(g_{i+1})\Phi_n(g_i) = e_i(e_i(e_i(e_i + e_{i+1}e_i) - 1 = \Phi_n(g_i)\Phi_n(g_{i+1})\Phi_n(g_i) = e_i(e_i(e_i - 1) + 2q) + qe_i - q(e_i(e_i + e_{i+1}e_i) - 1$$

since $q - q^2 + 2q = q$. \qed

**Remark 7.2** Let $A_\infty$ be the direct limit, $\lim_{\rightarrow} A_i$. It follows from the Markov relations that associate braids with links [5] that a sequence of traces $\phi_n : A_n \rightarrow k$ satisfying

$$\phi_{n+1}(x(g_{e_{n+1}} - 1)^{\pm 1}) = \phi_n(x) \quad \forall x \in A_n.$$ 

gives an invariant of links in $R^3$ under isotopy.

The question of Pimsner and Popa has been addressed algebraically in this paper. The extent that the present paper holds for operator algebras is under investigation by the author. One can phrase several of the properties of operator algebras in terms of module, such as amenability as a cohomological dimension
zero phenomenon. One can hopefully then define a relative amenability and split extension that would give a version of propositions 1.2 and 1.3 for operator algebras. The phrasing of the unitality condition is then hopefully clear, which should give an angle on several old and new theorems about shared properties of subalgebra and algebra. For example, the theorem of A. Connes that M is the hyperfinite factor iff N is hyperfinite (where M and N are $II_1$ factors with $N \subseteq M$ of finite index) should be reprovable in this way.

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