PSEUDO-DIFFERENTIAL PERTURBATIONS AND STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS:

Dirichlet feedback control problems.

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2) the boundary condition

3) both the operator equation and the boundary condition

and we prove the existence of feedback semigroups in these cases. The main tool in the investigation is a pseudo-differential transformation that transforms the domains of the feedback semigroup generators into classical operator domains, where standard resolvent analysis can be employed. The transformations turns out to be a shortcut to some of the stabilization results of I. Lasiecka and R. Triggiani in [7], [8] and [9], and it enlightens to some extent how a change of boundary condition influences the systems.
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ORIENTATION.

This paper is concerned with boundary feedback stabilization problems for parabolic and hyperbolic evolution equations associated with elliptic differential operators. Boundary feedback systems have been studied intensively during the last ten years, with a monumental contribution due to I. Lasiecka & R. Triggiani; we refer here only to [7], [8] and [9]. Lasiecka & Triggiani employ a semigroup-rooted method, as related to work of Washburn [15] and Balakrishnan [1]. This semigroup-integral representation method is based on the theory of fractional power spaces and second order operators are considered. We describe here a pseudo-differential operator method that allow us to work directly in \(L^2(\Omega)\) and we construct in a "classical" manner the resolvent of the \(2m\)-order operator of the evolution equation. The explicit resolvent construction is then used to derive stabilization results for the boundary feedback semigroup of the problem. More important, however, is the application of the pseudo-differential method to characterize various types of feedback systems appearing in the literature. Some of the first results on the stabilization of feedback systems are due to Nambu [11] and Triggiani [12], where the systems are stabilized by a suitable manipulation (control) in the operator equation. Such manipulations are covered by what we denote by perturbations of the first kind. More complicated than manipulating with the operator equation, is manipulation with the boundary condition. Stabilization of a system by change of boundary condition is treated in [7], [8] and [9]. It is this kind of manipulation with the boundary condition that is usually understood as a boundary control, or as a boundary feedback for systems where the control is a feedback of the state. We denote this kind of control a perturbation of the second kind. For both types of perturbations it is most realistic to assume that the feedback controls are of finite rank.

One of the main results of this paper is that we can, in general, transform a perturbation of the second kind into the more simple and well understood perturbation of the first kind, where a "classical" approach can be taken for obtaining stabilization results. It turns out that the transformation can be regarded as a generalized change of coordinates in the space \(H^m(\Omega)\), a feature that is also of interest when considering optimal control of systems, but this is not elaborated on in this paper.
For the sake of generality, we also introduce a mixing of the perturbations. We manipulate both the operator equation and the boundary condition and call this a perturbation of the third kind. For such systems it is possible to construct a system operator that is variational (i.e. associated with a suitable sesquilinear form), and from calculation of this operator the linking of "interior terms" and "boundary terms" is revealed. The stabilization procedure suggested for such systems with mixed feedbacks is probably not optimal in any sense, but it is simple, and the theory elucidates the nature of the boundary feedback systems.

The pseudo-differential approach thus allows us, in a unified setting, to obtain stabilization results for all three kinds of perturbations and for parabolic as well as hyperbolic equations. In all cases we conclude that it is possible to construct finite rank feedback mechanisms that give exponential decrease of the $L^2$-norm of the state. For perturbations of the first and second kind this is achieved by means of the pole-assignment theorem, while for perturbations of the third kind interior and boundary terms are balanced.

The results on perturbations of first and second kind are contained in § 5 of the paper, while § 6 deals with the perturbations of the third kind. § 1 is an introduction to the notation used throughout the paper, § 2 introduces the specific form of the perturbations considered. § 3 and § 4 deals with some results of pseudo-differential calculus, most importantly the transformation techniques employed.
§ 1. INTRODUCTION AND NOTATION.

We consider stabilization of parabolic and hyperbolic differential equations of the form

\[
\begin{cases}
\partial_t u + Au = 0 & \text{in } \Omega, \text{ for } t > 0 \\
u = 0 & \text{on } \Gamma, \text{ for } t > 0 \\
u = u_0 & \text{in } \Omega, \text{ for } t = 0 
\end{cases}
\]  
(1.1)

and

\[
\begin{cases}
\partial_t^2 u + Au = 0 & \text{in } \Omega, \text{ for } t \in \mathbb{R} \\
u = 0 & \text{on } \Gamma, \text{ for } t \in \mathbb{R} \\
u = u_0 & \text{in } \Omega, \text{ at } t = 0 \\
_t u = u_1 & \text{in } \Omega, \text{ at } t = 0 
\end{cases}
\]  
(1.2)

Here \( A \) is a formally selfadjoint, uniformly strongly elliptic differential operator of order \( 2m \), of the form

\[
A = \sum_{|\alpha|, |\beta| \leq m} D_\beta a_{\alpha \beta}(x) D^\alpha,
\]
(1.3)

with \( C^\infty(\bar{\Omega}) \)-coefficients \( a_{\alpha \beta} \) on a bounded, open domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \). with smooth boundary \( \partial \Omega = \Gamma \). \( \gamma \) is the Dirichlet trace operator

\[
\gamma = \{ \gamma_j \}_{0 \leq j < m}, \text{ where}
\]
(1.4)

\[
\gamma_j u = \left( \frac{1}{i} \frac{\partial}{\partial n} \right)^j u \bigg|_\Gamma
\]
(1.5)

(n is the normal, directed inward.)

We denote similarly

\[
\nu = \{ \nu_j \}_{0 < j \leq 2m},
\]
(1.6)

the Neumann trace operator, and we define the Cauchy-data \( \rho u \) as

\[
\rho u = \{ \rho u, uu \},
\]
(1.7)
moreover we use the multi-indexes notation

\[
D^{\alpha} = D_{1}^{\alpha_{1}} \ldots D_{n}^{\alpha_{n}}, \\
\alpha_{j} = (\frac{1}{i} \frac{\partial}{\partial x_j})^{\alpha_{j}}
\]  

(1.8)

The Dirichlet realization $A_\gamma$ of $A$ is the operator acting like $A$ in $L^2(\Omega)$, and with domain

\[
D(A_\gamma) = \{ u \in H^{2m}(\Omega) \mid \gamma u = 0 \} = H^{2m}(\Omega) \cap H_0^m(\Omega)
\]

(1.9)

where $H^s(\Omega)$ is the Sobolev space of $L^2(\Omega)$-functions with $L^2(\Omega)$-derivatives up to order $s$. It is well known that $A_\gamma$ is an unbounded, selfadjoint operator in $L^2(\Omega)$, and since the embedding $H^s(\Omega) \to H^t(\Omega)$ is compact for $s > t$, the resolvent $R(\lambda, A_\gamma)$ of $A_\gamma$ is a compact operator in $L^2(\Omega)$ for all $\lambda$ outside the spectrum, $\text{sp}(A_\gamma)$, of $A_\gamma$. Hence $A_\gamma$ has a sequence of real eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ converging to infinity. We see that (1.1) and (1.2) are the time dependent evolution problems associated with $A_\gamma$, generalizing the heat equation, resp. the wave equation. When $\lambda_1 > 0$, all solutions $u(t,x)$ of (1.1) are exponentially decreasing for $t \to \infty$, and all solutions of (1.2) are bounded; we will call this the stable case. However, if some eigenvalues are negative, there are solutions both of (1.1) and (1.2) that blow up in an exponential manner for $t \to \infty$. It is therefore of interest to investigate how one can change the systems to obtain the stable case, and this is the aim of this paper.
\section*{2. Perturbations of the Boundary Value Problems.}

By a \textit{perturbation of the first kind} of the system (1.1), we will understand

\begin{equation}
\begin{aligned}
\partial_t u + Au + Cu &= 0 \quad \text{in } \Omega, \text{ for } t > 0 \\
\gamma u &= 0 \quad \text{on } \Gamma, \text{ for } t > 0 \\
u &= u_0 \quad \text{in } \Omega, \text{ at } t = 0
\end{aligned}
\tag{2.1}
\end{equation}

Here the interior operator $A$ is replaced by $A + G$ where $G$ has finite rank
and is of the special form $G = KT$, where

(i) $T$ is a \textit{trace operator} that maps functions on $\Omega$ into functions on
$\Gamma$, of the form as a column vector

(ii) $K$ is a \textit{Poisson operator} that maps functions on $\Gamma$ into functions on
$\Omega$, of the form as a row vector

An operator $G$ of this special form is denoted a \textit{singular Green operator},
this kind of operator is carefully explained, as well as the trace- and
Poisson- operators, in Grubb [5]. They are all operators entering in the
"Boutet de Monvel-calculus" (c.f. Boutet de Monvel [2]).

Stabilization of the a priori unstable system (1.1) by a perturbation of
the first kind has been studied e.g. in Nambu [11] and Trigiani [14],
and it is shown there that it is possible to choose $G$ of finite rank,
such that (2.1) is stable.

By a \textit{perturbation of the second kind} of the system (1.1), we will under-
stand a system of the form

\begin{equation}
\begin{aligned}
\partial_t u + Au &= 0 \quad \text{in } \Omega, \text{ for } t > 0 \\
\gamma u &= T'u \quad \text{on } \Gamma, \text{ for } t > 0 \\
u &= u_0 \quad \text{in } \Omega, \text{ at } t = 0
\end{aligned}
\tag{2.2}
\end{equation}

where the boundary operator $\gamma$ is replaced by $\gamma - T'$, $T'$ being a trace
operator of finite rank.
Both (2.1) and (2.2) are so called boundary feedback systems, and we will especially be interested in the case where $T'$ is of the special form

\[(2.3)\quad T'u = \sum_{j=1}^{N} (u|w_j)g_j\]

(here $\langle \cdot | \cdot \rangle$ is the usual $L^2(\Omega)$ - inner product, $w_j \in C^\infty(\overline{\Omega})$, $g_j \in C^\infty(\Gamma)^m$, $j = 1, 2, \ldots, N$).

For applications, $w_j$ can be thought of as sensor functions and $g_j$ as boundary actuators.

$T'$ defined in (2.3) is called a finite dimensional feedback operator, and in contrast to $\tau$ it is of a non-local nature. (One of the major differences between the perturbations (2.1) and (2.2) is that in (2.1) the boundary condition is local, whereas in (2.2) it is non-local).

Boundary feedback systems have been studied in a number of papers by I. Lasiecka and R. Triggiani (see e.g. Lasiecka & Triggiani [7], [8] and [9]). One of the main results is that, under suitable hypotheses, it is possible to choose the functions $w_j$ and $g_j$ appearing in (2.3), such that the system is stable. Lasiecka and Triggiani take a semigroup approach to investigate the system (2.2), using developments on the semigroup approach presented in Washburn [15] and Balakrishnan [1]. (The basic idea of a semigroup model is presented in Fattorini [4], where ordinary differential equations are considered.)

By a perturbation of the third kind of the system (1.1), we will understand a system of the form

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t u + Au + Gu = 0 \quad \text{in } \Omega, \text{ for } t > 0 \\
\nu u = T'u \quad \text{on } \Gamma, \text{ for } t > 0 \\
u = u_0 \quad \text{in } \Omega, \text{ at } t = 0
\end{array} \right.
\end{align*}
\]

(2.4)

where the operators $G$ and $T'$ are of the types considered above.

We define perturbations of the system (1.2) analogously.
In the following we present a pseudo-differential operator method to investigate the systems above, this gives us in an easy way many of the results of Lasiecka & Triggiani, [7], [8] and [9], as well as similar results for the hyperbolic problem

\[
\begin{aligned}
\partial^2_t u + Au &= 0 \quad \text{in } \Omega, \quad \text{for } t \in \mathbb{R} \\
\gamma u - T'u &= 0 \quad \text{on } \Gamma, \quad \text{for } t \in \mathbb{R} \\
u &= u_0 \quad \text{in } \Omega, \quad \text{at } t = 0 \\
\partial_t u &= u_1 \quad \text{in } \Omega, \quad \text{at } t = 0
\end{aligned}
\]

(a perturbation of the second kind of (1.2)).

Here we would like to point out that there now exists a quite general theory that includes all the above perturbations when they have "smooth coefficients", namely the theory of pseudo-differential boundary problems. For these, the solvability of parabolic problems like (2.2) (and far more general cases) has been discussed in great detail in Grubb [5]. However, in the work that follows, we use only some basic results of the pseudo-differential methods, but the techniques are crucial for the simplicity of the proofs, and the theory is very helpful for the understanding of the underlying problems.

Our main result is that we can, in general, transform a perturbation of the second kind into a perturbation of the first kind, whenever the boundary condition is normal, in the sense described in Grubb [5]. This includes all "classical" normal boundary conditions, as well as the feedback boundary condition \(\gamma u - T'u = 0\), with \(T'\) given by (2.3). In this case, however, a special transformation of the systems (2.2) and (2.5) proves to be very useful. It turns out that the transformation can be regarded as a generalized change of coordinates, and the resulting, transformed system operator \(A + G\) is merely a finite dimensional, \(A\)-bounded perturbation of \(A\). In this case, stabilization theory for perturbations of the first kind is straightforward, as we can apply the well known "pole assignment theorem" due to Wonham (see Wonham [16]).
§ 3. NORMAL OPERATOR REALIZATIONS.

Assume that \( Tu = 0 \) is a normal boundary condition in the sense of Grubb [5]. We will then define a normal realization of the operator \( A \) (1.3) the following way: Let \( A_T \) be the operator acting like \( A \) in \( L^2(\Omega) \) with domain

\[
D(A_T) = \{ u \in H^m(\Omega) \mid Tu = 0 \}
\]

Then the realization \( A_T \) of \( A \) is called a normal realization.

It is shown in Grubb [5] Lemma 1.6.8 that normal realizations have dense domains in \( L^2(\Omega) \). According to Grubb, there exists an operator \( A \), (see (4.4)) which is a homeomorphism in \( H^s(\Omega) \) for any \( s \geq 0 \), such that \( A \) defines a bijection

\[
A : D(A_T) \sim D(A_T) = H^m(\Omega) \cap H^m_0(\Omega)
\]

where \( A_T \) is the Dirichlet realization from § 1.

Moreover, we have

**LEMMA 3.1**

Let \( T' \) be given by (2.3) and define

\[
T = \tau - T'
\]

Then \( Tu = 0 \) is a normal boundary condition, and the operator realization \( A_T \) of \( A \) is a closed, densely defined operator in \( L^2(\Omega) \).

**Proof:**

We only have to show that \( A_T \) is closed.

Let \( (u_n) \) be a sequence in \( D(A_T) \) converging to \( u \in L^2(\Omega) \), and assume that \( (Au_n) \) converges to \( v \in L^2(\Omega) \).

We must show that \( u \in D(A_T) \) with \( Au = v \).
Since $u_n \to u$ in $L^2(\Omega)$, we have that $Au_n \to Au$ in $D'(\Omega)$ (the dual space of $C_0^\infty(\Omega)$), so $v = Au \in L^2(\Omega)$ and $u_n \to u$ in the space $\{v \in L^2(\Omega) \mid Av \in L^2(\Omega)\}$. Now $\gamma u_n \to \gamma u$ in $\prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma)$ (see Lions & Magenes [10]) and since

$$\gamma u_n = \sum_{j=1}^{N} (u_n|w_j)g_j$$

where $(u_n|w_j) \to (u|w_j)$, $j = 1, 2, \ldots, N$.

we see that $\gamma u_n \to \sum_{j=1}^{N} (u|w_j)g_j$, so $\gamma u = \sum_{j=1}^{N} (u|w_j)g_j$ as an element of $\prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma)$. Now, since

$$g_j \in C_0^\infty(\Gamma)^m \subseteq \prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma), \quad j = 1, 2, \ldots, N.$$  

we have that

$$\gamma u \in \prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma).$$

But then by the regularity of the Dirichlet problem for $A$, $Au \in L^2(\Omega)$ and $\gamma u \in \prod_{0 \leq k < m} H^{2m-k-\frac{1}{2}}(\Gamma)$ imply that $u \in H^{2m}(\Omega)$.

Altogether, $u \in H^{2m}(\Omega)$ with $\gamma u - \sum_{j=1}^{N} (u|w_j)g_j = 0$, so $u \in D(A_T).$ \hfill \Box
§ 4: THE PSEUDO-DIFFERENTIAL TRANSFORMATIONS

Consider for \( \ell = 1, 2 \) the parabolic, resp. hyperbolic perturbation of the second kind

\[
\partial_t^\alpha u + A_T u = 0, \quad u \in D(A_T)
\]

with \( T = \gamma - T' \), \( T' \) given by (2.3).

Using (3.2) this can be transformed into

\[
\partial_t^\alpha A^{-1} v + A_T A^{-1} v = 0, \quad v \in D(A_T)
\]

where \( v = \Lambda u \). Acting with \( \Lambda \) from the left in (4.2) we find

\[
\partial_t^\alpha v + \Lambda A_T A^{-1} v = 0, \quad v \in D(A_T).
\]

From Grubb [5], Lemma 1.6.8, the structure of \( \Lambda \) and \( \Lambda^{-1} \) are as follows

\[
\begin{align*}
\Lambda &= 1 - K_\delta T' \\
\Lambda^{-1} &= 1 - K_\delta Q_\delta T'
\end{align*}
\]

where \( K_\delta, \delta \in [0, \delta_0] \) is a family of Poisson operators and \( Q_\delta \) is a certain \( \delta \)-dependent pseudo-differential operator, that is bijective and elliptic in \( \bigcap_{0 \leq k \leq 2m} H^{s-k}(\Omega), \ s \geq 0 \).

Then

\[
\Lambda A_T A^{-1} = (1 - K_\delta T')(1 - K_\delta Q_\delta T') = A_T + G
\]

where

\[
G = K_\delta T' A_T K_\delta Q_\delta T' - A_T K_\delta Q_\delta T' - K_\delta T' A_T
\]

is a singular Green operator of finite rank.

Hence (4.3) is a perturbation of the first kind.
However in the case, where $T'$ is of the special form (2.3), we can use a more explicit transformation.

Assume, for the moment, that $O$ is not an eigenvalue of $A_\gamma$. This is only a temporary assumption that will be removed later (see after remark 5.3). Let $K_\gamma$ be the Poisson operator that solves the Dirichlet problem for $A$, i.e. $u = K_\gamma \varphi$ is the solution of

$$
\begin{align*}
Au &= 0 \text{ in } \Omega \\
\gamma u &= \varphi \text{ on } \Gamma
\end{align*}
$$

Since $K_\gamma T'$ has finite rank, the bounded operator $1 - K_\gamma T'$ is a Fredholm operator with index 0 in $L^2(\Omega)$, and it maps $H^{2m}(\Omega)$ into $H^{2m}(\Omega)$. (cf. Hörmander [6], ch. 19.1 for a simple explanation of Fredholm operators like $1 - K_\gamma T'$)

Since

$$
\gamma(1 - K_\gamma T')u = \gamma u - T'u = Tu
$$

we have that

$$
(1 - K_\gamma T')D(A_T) \subseteq D(A_\gamma)
$$

and moreover, if $u \in D(A_T)$ and $v = (1 - K_\gamma T')u$, then

$$
Av = A(1 - K_\gamma T')u = Au
$$

Now assume that $T'$ can be chosen such that $1 - K_\gamma T'$ is a bijection in $H^{2m}(\Omega)$ (this will be done later on, see § 5). Then referring again to the theory of Fredholm operators, we have that $1 - K_\gamma T'$ defines a homeomorphism:

$$
1 - K_\gamma T' : D(A_T) \sim D(A_\gamma) \quad (= H^{2m}(\Omega) \cap H^{m}_0(\Omega))
$$

just like the $A$-operator above.

We have then established the useful factorization

$$
A_T = A_\gamma (1 - K_\gamma T')
$$
Now proceeding as above, the problems

\[(4.13) \quad \partial_t^\ell u + A_T u = 0 \quad , \quad u \in D(A_T) \quad , \quad \ell = 1, 2\]
	ransforms into

\[(4.14) \quad \partial_t^\ell v + (1 - K_T T') A_\gamma v = 0 \quad , \quad v \in D(A_\gamma) \quad , \quad \ell = 1, 2\]

where \( v = (1 - K_T T') u \).

Thus we have transformed the perturbation of the second kind \((4.13)\) into a perturbation of the first kind \((4.14)\), and we are able to calculate the system operator in an easy way.

We have

**THEOREM 4.1**

The boundary feedback systems

\[
(4.15) \quad \begin{cases} 
\partial_t u + A u = 0 & \text{in } \Omega \quad , \quad \text{for } t > 0 \\
\gamma u = T'u & \text{on } \Gamma \quad , \quad \text{for } t > 0 \\
u = u_0 & \text{in } \Omega \quad , \quad \text{at } t = 0
\end{cases}
\]

and

\[
(4.16) \quad \begin{cases} 
\partial_t^2 u + A u = 0 & \text{in } \Omega \quad , \quad \text{for } t \in \mathbb{R} \\
\gamma u = T'u & \text{on } \Gamma \quad , \quad \text{for } t \in \mathbb{R} \\
u = u_0 & \text{in } \Omega \quad , \quad \text{at } t = 0 \\
\partial_t u = u_t & \text{in } \Omega \quad , \quad \text{at } t = 0
\end{cases}
\]

with \( T' \) given by \((2.3)\), transform into the systems

\[
(4.15') \quad \begin{cases} 
\partial_t v + A v - K_T T' A v = 0 & \text{in } \Omega \quad , \quad \text{for } t > 0 \\
\gamma v = 0 & \text{on } \Gamma \quad , \quad \text{for } t > 0 \\
v = v_0 & \text{in } \Omega \quad , \quad \text{for } t = 0
\end{cases}
\]
and

\[
\begin{cases}
\partial_t^2 v + Av - K_\gamma T'Av = 0 \text{ in } \Omega, \text{ for } t \in \mathbb{R} \\
\gamma v = 0 \text{ on } \Gamma, \text{ for } t \in \mathbb{R} \\
v = v_0 \text{ in } \Omega, \text{ for } t = 0 \\
\partial_t v = v_1 \text{ in } \Omega, \text{ for } t = 0
\end{cases}
\]

(4.16')

Since

\[
K_\gamma T'Av = \sum_{j=1}^{N} (Av[w_j])K_\gamma g_j
\]

(4.17)

for \( v \in H^{2m}(\Omega) \), \( K_\gamma T' A \) has finite rank, and we see that

\[
\tilde{\Lambda} = \Lambda - K_\gamma T' A
\]

(4.18)

can be regarded as a finite dimensional perturbation of \( \Lambda \).

We obviously have (\( ||\cdot||_s \) is the \( H^s(\Omega) \)-norm):

\[
||K_\gamma T'Av||_0 \leq c||Av||_0 \leq c||Av||_0 + ||v||_0
\]

(4.19)

for \( v \in D(A_\gamma) \), so \( K_\gamma T' A \) is \( \Lambda \)-bounded. Since \( A_\gamma \) is the infinitesimal generator of an analytic semigroup on \( L^2(\Omega) \), so is \( \tilde{\Lambda}_\gamma \), form the perturbation result in Zabczyk [17], prop. 1.

We have

**Theorem 4.2**

The realization \( \tilde{\Lambda}_\gamma \) of the operator

\[
\tilde{\Lambda} = \Lambda - K_\gamma T' A
\]

(4.20)

with domain
\[(4.21) \quad D(\tilde{A}_\gamma) = H^{2m}(\Omega) \cap H^m_0(\Omega) \quad (= D(A_\gamma))\]

is the infinitesimal generator of an analytic semigroup \(e^{-A_\gamma t}\), \(t \geq 0\) on \(L^2(\Omega)\), giving the solution to (4.15') as

\[(4.22) \quad v(t,x) = e^{-A_\gamma t} v_0(x), \quad x \in \Omega, \quad t \geq 0.\]

when \(v_0 \in L^2(\Omega)\). The solution to the original system (4.15) is then

\[(4.23) \quad u(t,x) = (1 - K_{\gamma T'})^{-1} e^{-A_\gamma t} (1 - K_{\gamma T'})u_0(x), \quad x \in \Omega, \quad t \geq 0\]

when \(u_0 \in L^2(\Omega)\).

**Remark 4.3**

All the semigroup theory used above is covered by [5].
§ 5. AN APPLICATION OF THE PSEUDO-DIFFERENTIAL TRANSFORMATION TO STABILIZATION.

We will now show how the transformation from § 4 can be used as a shortcut to some of the results of Lasiecka & Triggiani ([7],[8],[9]), which have been a great motivation to us.

The assumed instability of the systems (1.1) and (1.2) is caused by the negative eigenvalues in the pure point spectrum \( \text{sp}(A_\gamma) \) of \( A_\gamma \), and we will show that we can choose a finite dimensional feedback boundary condition

\[
(5.1) \quad \gamma u = T'u
\]

where \( T' \) is defined by

\[
(5.2) \quad T'u = \sum_{j=1}^{N} (u | w_j) g_j
\]

(see (2.4)), such that the systems

\[
(5.3) \begin{cases}
\partial_t u + Au = 0 & \text{in } \Omega, \text{ for } t > 0 \\
\gamma u = T'u & \text{in } \Gamma, \text{ for } t > 0 \\
u = u_0 & \text{in } \Omega, \text{ for } t = 0
\end{cases}
\]

and

\[
(5.4) \begin{cases}
\partial_t^2 u + Au = 0 & \text{in } \Omega, \text{ for } t \in \mathbb{R} \\
\gamma u = T'u \text{ on } \Gamma, \text{ for } t \in \mathbb{R} \\
u = u_0 & \text{in } \Omega, \text{ at } t = 0 \\
\partial_t u = u_t & \text{in } \Omega, \text{ at } t = 0
\end{cases}
\]

are stable systems (in the sense described in § 1). We will apply the pseudo-differential transformation to the perturbations of the second kind (5.3) and (5.4) and then apply Wonhams "pole assignment theorem" (Wonham [16]) on the resulting perturbations of the first kind. This, combined with a classical resolvent analysis, gives us the desired results.
Let the eigenvalues of $A_\gamma$ be arranged in a non-decreasing sequence

\begin{equation}
\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{K-1} \leq 0 \leq \lambda_K \leq \ldots
\end{equation}

each eigenvalue repeated according to multiplicity, and let $\{\varphi_j\}_{j \geq 1}$ be a corresponding set of orthonormalized eigenfunctions of $A_\gamma$. Now define $P_u$ and $P_s$ as the orthogonal projections of $L^2(\Omega)$ on the orthogonal subspaces $X_u$, resp. $X_s$, defined by

\begin{equation}
\begin{cases}
X_u = \text{span}\{\varphi_j\}_{1 \leq j < K} \\
X_s = \text{span}\{\varphi_j\}_{j \geq K}
\end{cases}
\end{equation}

Remark 5.1

The results in Lasiecka & Triggiani [7], [8] and [9] are formulated as if non-selfadjoint realizations are treated as well, but on the other hand the treatment is based heavily on the orthogonal projections on the eigenspaces $X_u$ and $X_s$. Orthogonality of eigenspaces in general requires at least that $A_\gamma$ is normal, i.e. $A_\gamma^* A_\gamma = A_\gamma A_\gamma^*$, but we know of no Dirichlet realization $A_\gamma$ satisfying this without being selfadjoint.\)

Since $X_u$ and $X_s \cap D(A_\gamma)$ are invariant subspaces for $A_\gamma$, we can define the restrictions

\begin{equation}
\begin{cases}
A_u = A_\gamma|_{X_u} \\
A_s = A_\gamma|_{X_s \cap D(A_\gamma)}
\end{cases}
\end{equation}

Then $A_u$ is a bounded operator on $X_u$ and $A_s$ is an unbounded operator with domain $D(A_s) = X_s \cap D(A_\gamma)$. Notice that $P_u$ and $P_s$ commute with $A_\gamma$ on $D(A_\gamma)$. Now writing $f_u = P_u f$, $f_s = P_s f$ for $f \in L^2(\Omega)$, we have that when $u \in D(A_\gamma)$, (see 3.1) and (4.11)), then $v = (1-K_\gamma T^\prime)u \in D(A_\gamma)$ satisfies

$$Av = Au$$

\begin{equation}
\begin{cases}
v_u \in X_u \\
v_s \in D(A_s)
\end{cases}
\end{equation}
Now we use the factorization

\[(5.9) \quad A_T = A_\gamma (1 - K_\gamma T')\]

in the discussion of the resolvent equation

\[(5.10) \quad (A_T - \lambda)u = f, \quad f \in L^2(\Omega)\]

First we consider the case where we are allowed to decouple the feedback by assuming that

\[(5.11) \quad P_s w_j = 0, \quad j = 1, 2, \ldots, N\]

(i.e. the \(w_j\) are in \(X_u\): the "unstable" eigenspace).

Then we write (5.10) in projected and factorized form

\[
(5.12) \quad \begin{pmatrix} \begin{pmatrix} P_u \\ P_s \end{pmatrix} \end{pmatrix} (A_\gamma (1 - K_\gamma T') (u_u + u_s) - \lambda (u_u + u_s) = \begin{pmatrix} f_u \\ f_s \end{pmatrix} \]

which because of (5.11) reduces to

\[
(5.13) \quad A_u u_u - A_P K T' u_u - \lambda u_u = f_u
\]

\[
(5.14) \quad - A_s P K T' u_u + A_s u_s - \lambda u_s = f_s
\]

where we observe that (5.13) is a finite dimensional resolvent equation for the matrix operator

\[(5.15) \quad \overline{A_u} = A_u - A_P K T'\]

To this we can apply Wonhams theorem to stabilize the unstable part of the system. We then have (for \(m = 1\), one of Lasiecka and Triggiani's results):

**Theorem 5.2**

Assume that the Neumann traces \(\{w_j\}\) are linearly independent, so that

\[(5.16) \quad \dim (uX_u) = \dim (X_u) (= K - 1)\]
and let \( \{c_j\}_{1 \leq j < K} \) be an arbitrary given set of \( K - 1 \) distinct, real numbers.

Then there exists a number \( N \) and a set

\[
\{w_j, g_j\}_{1 \leq j \leq N}
\]

where \( w_j \in X_u \) and \( g_j \in C^\infty(\Gamma)^m \), such that with

\[
T'u = \sum_{j=1}^{N} (u|w_j)g_j
\]

the eigenvalues of the matrix operator

\[
\overline{A}_u = A_u - A_u P K T'
\]

on \( X_u \) are \( \{c_j\}_{1 \leq j < K} \).

The number \( N \) can be taken as the largest multiplicity of the unstable eigenvalues \( \{\lambda_j\}_{1 \leq j < K} \). In particular, \( N = 1 \) when the eigenvalues are simple.

**Proof:**

Assume first that all of the eigenvalues \( \{\lambda_j\}_{1 \leq j < K} \) are simple and take \( N = 1 \). Consider \( T' \) of the form

\[
T'u = (u|w)g
\]

In the basis \( \{\varphi_j\}_{1 \leq j < K} \) of \( X_u \), the matrix \( \overline{A}_u \) has the form

\[
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & & \\
& & \ddots & \\
0 & & & \lambda_{K-1}
\end{pmatrix} - P(g)W^t
\]
where

\[
\begin{pmatrix}
\lambda_1(K_1 g | \varphi_1) \\
\lambda_2(K_1 g | \varphi_2) \\
\vdots \\
\lambda_{K-1}(K_1 g | \varphi_{K-1})
\end{pmatrix}
\]

and \( W^t \) is the transposed of the matrix given by

\[
W = \begin{pmatrix}
(\varphi_1 | w) \\
(\varphi_2 | w) \\
\vdots \\
(\varphi_{K-1} | w)
\end{pmatrix}
\]

Consider now the control matrix of the pair \((A_u, W)\):

\[
\begin{pmatrix}
W & A_u & \ldots & A_u^{K-2} & W
\end{pmatrix} = \begin{pmatrix}
(\varphi_1 | w) & \lambda_1(\varphi_1 | w) & \ldots & \lambda_1^{K-2}(\varphi_1 | w) \\
(\varphi_2 | w) & \lambda_2(\varphi_2 | w) & \ldots & \lambda_2^{K-2}(\varphi_2 | w) \\
\vdots & \vdots & \ddots & \vdots \\
(\varphi_{K-1} | w) & \lambda_{K-1}(\varphi_{K-1} | w) & \ldots & \lambda_{K-1}^{K-2}(\varphi_{K-1} | w)
\end{pmatrix}
\]

The determinant of the control matrix is computed to be

\[
\prod_{1 \leq j < K} (\varphi_j | w) \prod_{1 \leq \ell < \delta < K} (\lambda_\delta - \lambda_\ell)
\]

by reduction to a Vandermonde determinant. Since the eigenvalues are assumed to be simple, we can choose \( w \in X_u \), satisfying \( (\varphi_j | w) \neq 0 \) for \( j = 1, 2, \ldots, K-1 \), such that the determinant is different from 0. This implies that the pair \((A_u, W)\) is controllable, so by the pole assignment theorem there exists a matrix

\[
\tilde{P} = \begin{pmatrix}
P_1 \\
P_2 \\
P_{K-1}
\end{pmatrix}, \quad p_j \in \mathbb{C}, \quad j = 1, 2, \ldots, K-1
\]

for which the matrix \( A_u - \tilde{P} W^t \) has the set \( \{c_j\}_{1 \leq j < K} \) as eigenvalues.

Now we will choose \( g \in C^\infty(\Gamma)^m \) such that \( \tilde{P} = P(g) \), i.e. such that

\[
\lambda_j(K_1 g | \varphi_j) = p_j
\]
for \( j = 1, 2, \ldots, K - 1 \).

Now from the formula (A.10) in the appendix we have that

\[
(5.28) \quad (K, g | \psi_j) = \frac{-1}{\lambda_j} (g | \mathcal{A}^{10}_m \psi_j)_\Gamma.
\]

Here \( \mathcal{A}^{10}_m \) is (since \( A \) is elliptic) an invertible \( m \times m \) matrix of differential operators over \( \Gamma \), that maps \( \prod_{m \leq k < 2m} H^{2m-k-1/2}(\Gamma) \) onto \( \prod_{0 \leq k < m} H^{m+k+1/2}(\Gamma) \), and \( (\cdot | \cdot)_\Gamma \) is the \( L^2(\Gamma)^m \) inner product.

Since the set

\[
(5.29) \quad \{ \psi_1, \psi_2, \ldots, \psi_{K-1} \}
\]

is linearly independent, so is also the set

\[
(5.30) \quad \{ \mathcal{A}^{10}_m \psi_1, \mathcal{A}^{10}_m \psi_2, \ldots, \mathcal{A}^{10}_m \psi_{K-1} \}.
\]

Hence it is possible to choose \( g \in C^\infty(\Gamma)^m \), satisfying

\[
(5.31) \quad (g | \mathcal{A}^{10}_m \psi_j) = -p_j, \quad j = 1, 2, \ldots, K - 1
\]

and this choice of \( g \) provides us with the desired \( P(g) \), in view of (5.28). This ends the proof in the case of simple eigenvalues.

Now assume that one or more of the eigenvalues \( \{ \lambda_j \}_{1 \leq j \leq K} \) have multiplicity larger than 1, and let us take \( \sigma \) to be the largest occurring multiplicity. Take \( N = \sigma \) and consider \( T' \) of the form

\[
(5.32) \quad T'u = \sum_{j=1}^\sigma (u | w_j)g_j
\]
Now $A_u - A_u P K T'$ can be written

\[
\begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & & \\
& & \ddots & \\
0 & & & \lambda_{K-1}
\end{bmatrix} = P(\{g_i\}_{1 \leq i \leq \sigma}) W_t
\]

where

\[
P(\{g_i\}_{1 \leq i \leq \sigma}) = \begin{bmatrix}
\lambda_1(K, g_1 | \varphi_1) & \lambda_1(K, g_2 | \varphi_1) & \ldots & \lambda_1(K, g_\sigma | \varphi_1) \\
\lambda_2(K, g_1 | \varphi_2) & \lambda_2(K, g_2 | \varphi_2) & \ldots & \lambda_2(K, g_\sigma | \varphi_2) \\
\vdots & & \ddots & \\
\lambda_{K-1}(K, g_1 | \varphi_{K-1}) & \lambda_{K-1}(K, g_2 | \varphi_{K-1}) & \ldots & \lambda_{K-1}(K, g_\sigma | \varphi_{K-1})
\end{bmatrix}
\]

and

\[
W_\sigma = \begin{bmatrix}
(\varphi_1 | w_1) & (\varphi_1 | w_2) & \ldots & (\varphi_1 | w_\sigma) \\
(\varphi_2 | w_1) & (\varphi_2 | w_2) & \ldots & (\varphi_2 | w_\sigma) \\
\vdots & & & \\
(\varphi_{K-1} | w_1) & (\varphi_{K-1} | w_2) & \ldots & (\varphi_{K-1} | w_\sigma)
\end{bmatrix}
\]

Considering the form of the control matrix

\[
\begin{bmatrix}
W_\sigma & A_u W_\sigma & \ldots & A_u^{K-2} W_\sigma
\end{bmatrix}
\]

we see that if $w_1, w_2, \ldots, w_\sigma$ are chosen in $X_u$ such that

\[
\text{rank } W_\sigma = \sigma
\]

then the rank of the control matrix (5.36) is $K - 1$ (because a regular $(K-1) \times (K-1)$ submatrix can be extracted, after suitable row-column operations). Then according to Wonhams theorem there exists a complex matrix

\[
\tilde{P}_\sigma = \begin{bmatrix}
P_{11} & P_{12} & \ldots & P_{1\sigma} \\
P_{21} & P_{22} & \ldots & P_{2\sigma} \\
\vdots & \vdots & \ddots & \vdots \\
P_{K-1,1} & P_{K-1,2} & \ldots & P_{K-1,\sigma}
\end{bmatrix}
\]

such that the eigenvalues of the matrix $A_u - \tilde{P}_\sigma W_t$ are $\{c_j\}_{1 \leq j \leq K}$. 

Now (5.28) takes the form

\[(5.39) \quad (K_\gamma g_i | \phi_j) = \frac{-1}{\lambda_j} (g_i | d^{10\nu} \nu \phi_j)_{\Gamma}.\]

In view of (5.16) we can choose \( g_i \in C^0(\Gamma)^m \), \( i = 1, 2, \ldots, \sigma \) satisfying

\[(5.40) \quad (g_i | d^{10\nu} \nu \phi_j)_{\Gamma} = - p_{j1}, \quad j = 1, 2, \ldots, \ K-1; \quad 1 \leq i \leq \sigma, \]

and this choice of \( \{g_i\} \) provides us with the desired \( P(\{g_i\}_{1 \leq i \leq \sigma}) \).

**Remark 5.3**

For the application of the Wonham theorem here, it is important that the range of \( P_u K_\gamma \) fills out all of \( X_u \); this is reformulated as the question whether the Neumann traces of the Dirichlet eigenfunctions in \( X_u \) are linearly independent. In that case the results are easy to formulate and allow \( N \) to be very low, otherwise the results become increasingly complicated and require (in general) higher \( N \), the more linear dependence there is. Lasiecka & Triggiani have in [8] an upper and lower bound on the number \( N \) of feedback terms necessary in (5.18), once the number of linearly independent Neumann traces are given, but the discussion of the size of \( \dim(uX_u) \) for differential operators on general domains in \( \mathbb{R}^n \) in the literature is far from complete, as far as we know.

Let us now dispose of the temporary assumption \( 0 \in \text{sp}(A_\gamma) \). Everything done this far can without changes be done with the translated operator \( A_\gamma - \delta \), \( \delta > 0 \), so if \( 0 \in \text{sp}(A_\gamma) \), we take a \( \delta > 0 \) such that \( 0 \in \text{sp}(A_\gamma - \delta) \). Then after determining the new realization \( A_T - \delta \) that moves the eigenvalues \( \lambda_1 - \delta, \lambda_2 - \delta, \ldots, \lambda_{k-1} - \delta \), and leaves \( \lambda_j - \delta \), \( j \geq K \) unaffected, we just add \( \delta \) to \( A_T - \delta \). Then the moved eigenvalues will be increased with \( \delta \), and so will the unaffected eigenvalues, i.e. they will return to be the eigenvalues of \( A_s \).

Let us now choose the set \( \{c_j\} \) occurring in Theorem 5.2 such that \( c_j \geq \lambda_K (> 0), \ j = 1, 2, \ldots, K-1 \).

With \( T' \) chosen according to the theorem, the operator \( P_u (1 - K_\gamma T') \) is injective, hence bijective, on \( X_u \), and since \( w_j \in X_u \), \( 1 - K_\gamma T' \) is
the identity on \( X_s \). Then, as promised in § 4, \( 1 - K_T' \) is bijective from \( H^{2m}(\Omega) \) to \( H^{2m}(\Omega) \) \((K_T' \) has \( C^\infty \)-range) and maps \( D(A_\gamma) \) onto \( D(A_\gamma) \). This justifies the factorization (4,12).

Define \( R(\lambda, \overline{A}_u) \) as the resolvent of \( \overline{A}_u \) in \( X_u \), for all \( \lambda \notin \{ \lambda_j \}_{1 \leq j \leq K} \) and let \( R(\lambda, A_s) \) be the resolvent of \( A_s \) in \( X_s \), defined for all \( \lambda \notin \{ \lambda_j \}_{j \geq K} \).

We write the solution to (5,13) as

\[
(5.41) \quad u_u = R(\lambda, \overline{A}_u)f_u, \quad \lambda \notin \{ \lambda_j \}_{1 \leq j \leq K}
\]

Using that if \( u \in D(A_\gamma) \), then \( v = (1 - K_T')u \in D(A_\gamma) \) satisfies

\[
(5.42) \quad v_s = P_s(1 - K_T')u = P_s(1 - K_T')(u_u + u_s)
\]

\[
= u_s - P_{s \gamma} K_T' u_u \in D(A_s)
\]

and we write (5,14) as

\[
(5.43) \quad (A_s - \lambda)(u_s - P_{s \gamma} K_T' u_u) = f_s + \lambda P_{s \gamma} K_T' u_u.
\]

Then, for all \( \lambda \notin \{ \lambda_j \}_{j \geq K} \)

\[
(5.44) \quad u_s - P_{s \gamma} K_T' u_u = R(\lambda, A_s)(f_s + \lambda P_{s \gamma} K_T' u_u).
\]

Inserting (5,41), we find for all \( \lambda \notin \{ \{c_j \}_{1 \leq j \leq K} \cup \{ \lambda_j \}_{j \geq K} \} \):

\[
(5.45) \quad u_s = P_{s \gamma} K_T' R(\lambda, \overline{A}_u)f_u + R(\lambda, A_s)(f_s + \lambda P_{s \gamma} K_T' R(\lambda, \overline{A}_u)f_u)
\]

so that

\[
(5.46) \quad u = u_u + u_s
\]

\[
= R(\lambda, \overline{A}_u)f_u + P_{s \gamma} K_T' R(\lambda, \overline{A}_u)f_u u + R(\lambda, A_s)f_s + \lambda R(\lambda, A_s) P_{s \gamma} K_T' R(\lambda, \overline{A}_u)f_u
\]

\[
= (1 + P_{s \gamma} K_T' + \lambda R(\lambda, A_s) P_{s \gamma} K_T') R(\lambda, \overline{A}_u)f_u + R(\lambda, A_s)f_s.
\]
We see that we have

**Lemma 5.3**

The resolvent $R(\lambda, A_T)$ solving (5.13) - (5.14) is

\[
(5.47) \quad R(\lambda, A_T)f = \begin{pmatrix} f_u \\ f_s \end{pmatrix} = (R_{11} \quad R_{12}) \begin{pmatrix} f_u \\ f_s \end{pmatrix}
\]

where

\[
(5.48) \quad R_{11} = (1 + P_{s\gamma}T + \lambda R(\lambda, A_s)P_{s\gamma}T') R(\lambda, A_u)
\]

\[
R_{12} = R(\lambda, A_s)
\]

so $R(\lambda, A_T)$ is well defined for all $\lambda$ outside the spectrum.

\[\{c_j\} \cup \{\lambda_j\}_{j \geq K}, \text{ of } A_T, \text{ and maps } L^2(\Omega) \text{ into } H^{2m}(\Omega).\]

**Lemma 5.4**

The resolvent $R(\lambda, A_T)$ satisfies the inequality

\[
(5.49) \quad ||R(\lambda, A_T)||_{L^2,L^2} \leq \frac{c}{\text{dist}(\lambda, \text{co}(\text{sp}(A_T)))}
\]

as an operator in $L^2(\Omega)$. Here $c > 0$ is a constant independent of $\lambda$, and $\text{co}(\text{sp}(A_T))$ is the convex hull of the spectrum of $A_T$.

**Proof:**

Note that $\text{co}(\text{sp}(A_T)) = [\lambda_K, \infty[$.

$A_s$ is a selfadjoint, positive operator on $X_s \cap D(A_T)$, satisfying:
\[ \| (A_s - \lambda) u \|_0 \| u \|_0 \geq \| (A_s - \lambda) u \|_0 \]

\[ = \| (A_s - \text{Re} \lambda) u \|_0 - i \text{Im} \lambda \| u \|_0^2 \]

\[ = (\| (A_s - \text{Re} \lambda) u \|_0^2 + (i \text{Im} \lambda \| u \|_0^2) \)^{1/2} \]

\[ \geq \left\{ \begin{array}{ll}
| \text{Im} \lambda | \| u \|_0^2 & \text{if } \text{Re} \lambda \geq \lambda_K \\
(\| \lambda_K - \text{Re} \lambda \|_0^2 + (\| \lambda \|_0^2)^{1/2} \| u \|_0^2 & \text{if } \text{Re} \lambda \leq \lambda_K
\end{array} \right. \]

\[ \geq \text{dist} (\lambda, \text{co}(\text{sp}(A_T))) \| u \|_0^2 \] hence

\[ \| R(\lambda, A_s) \|_{L^2, L^2} \leq (\text{dist} (\lambda, \text{co}(\text{sp}(A_T))))^{-1}. \]

\[ R(\lambda, \tilde{A}_u) \] is a \((K - 1) \times (K - 1)\) matrix of the form

\[ R(\lambda, \tilde{A}_u) = (\text{det}(\tilde{A}_u - \lambda))^{-1} \cdot p(\lambda, \tilde{A}_u) \]

where \( p(\lambda, \tilde{A}_u) \) is a polynomial in \( \lambda \) (this follows easily from the inversion formula for matrices), and

\[ \text{det}(\tilde{A}_u - \lambda) = \text{constant} \cdot \prod_{1 \leq j < K} (\lambda - c_j). \]

Therefore \( \| R(\lambda, \tilde{A}_u) \|_{L^2, L^2} \) is \( O(|\lambda - c_j|^{-1}) \) in a neighbourhood of \( c_k \in \{c_j\}_{1 \leq j < K} \). For \( |\lambda| \to \infty \) we write

\[ \tilde{A}_u - \lambda = -\lambda(1 - \lambda^{-1} \tilde{A}_u) \]

and we see that \( R(\lambda, \tilde{A}_u) \) is \( O(|\lambda|^{-1}) \) for \( |\lambda| \to \infty \), since \( (1 - \lambda^{-1} \tilde{A}_u) \) can be inverted by a Neumann series for \( |\lambda| > \| \tilde{A}_u \|_{L^2, L^2} \). We can then conclude that

\[ \| R(\lambda, \tilde{A}_u) \|_{L^2, L^2} \leq M(\text{dist}(\lambda, \text{co}(\text{sp}(A_T))))^{-1} \]

where \( M \) is a constant, independent of \( \lambda \).
Obviously $\co(\text{sp}(\bar{A}_u)) \subseteq \co(\text{sp}(A_T))$, and since $\text{sp}(\bar{A}_u)$ is bounded, $||\lambda R(\lambda, \bar{A}_u)||_{L^2,L^2}$ is $O(1)$ for $|\lambda| \to \infty$; moreover, $P_sK_{\gamma}T'$ is bounded and the decomposition $f \to f_u + f_s$ is bounded and fixed. Then, from the form of $R(\lambda, A_T)$ (5.47) - (5.48), we find that

\[
(5.55) \quad ||R(\lambda, A_T)||_{L^2,L^2} \leq \frac{M'}{\text{dist}(\lambda, \co(\text{sp}(\bar{A}_u)))} + \frac{1}{\text{dist}(\lambda, \co(\text{sp}(A_T)))}
\]

\[
\leq \frac{c}{\text{dist}(\lambda, \co(\text{sp}(A_T)))} \quad \Box
\]

Using Lemma 3.1, Theorem 5.2 and Lemma 5.4 we find

**THEOREM 5.5**

There exists a finite dimensional boundary condition

\[(5.56) \quad \gamma u = T'u \quad \text{on } \Gamma\]

where

\[(5.57) \quad T'u = \sum_{j=1}^{N} (u|w_j)g_j\]

$w_j \in X_u$, $g_j \in C^0(\Gamma)^m$, $j = 1, 2, \ldots, N$, such that the realization $A_T$ of $A_T$ with domain

\[(5.58) \quad D(A_T) = \{u \in H^{2m}(\Omega)| Tu = \gamma u - T'u = 0\}\]

is the infinitesimal generator of an analytic semigroup $e^{-A_T^t}$, $t \geq 0$ on $L^2(\Omega)$, giving the solution to the Dirichlet boundary feedback parabolic system (5.3) as

\[(5.59) \quad u(t,x) = e^{-A_T^t} u_0(x), \quad x \in \Omega, \quad t \geq 0.\]
when $u_0 \in L^2(\Omega)$. The solution (5.59) satisfies the damping estimate

$$\|u(t,\cdot)\|_0 \leq M e^{-\lambda_k t} \|u_0\|_0, \quad t \geq 0, \quad M > 0,$$

where $\lambda_k$ is the first positive Dirichlet eigenvalue of $A$. Moreover, the operators

$$\cos(A_T^{1/2} t) \quad \text{and} \quad \sin(A_T^{1/2} t)$$

are well defined, and we can write the solution to the hyperbolic problem (5.4) as

$$u(t,x) = \cos(A_T^{1/2} t)u_0 + A_T^{-1/2} \sin(A_T^{1/2} t)u_1(x)$$

$x \in \Omega$, $t \in \mathbb{R}$, when $u_0, u_1 \in L^2(\Omega)$. □

Remark 5.6

One of the slightly mysterious facts about the perturbation of the second kind, is that the operator $A_T$ can never be semibounded (i.e. satisfy an inequality

$$\Re e^{i\theta}(A_T u | u) \geq c \|u\|_0^2$$

for some $c$ and $\theta$), when $T' \neq 0$. This is a consequence of Proposition 1.7.11 in Grubb [5], in particular, $A_T$ can never be selfadjoint. Since $A_T$ is never semibounded, the semigroup $e^{-A_T t}$, $t \geq 0$ is never a contraction semigroup, hence the constant $M$ in (5.60) is always greater than 1. This is also noticed by Lasiecka & Triggiani, who consider the translated Laplacian in [8]. This phenomenon is not encountered in the Neumann feedback Problem. □

Remark 5.7

Comparison of eqs. (5.59) and (4.23) show that for the semigroups we have

$$e^{-A_t} = (1 - K_T T')^{-1} e^{-\tilde{\gamma}(1 - K_T T')}.$$

(5.64)
This justifies the term "generalized change of coordinates" from § 2.0

Now it is straightforward to extend the theory to include more general cases where \( P_s w_j \neq 0 \). The operator \( T' \) considered above can be written

\[
T'u = \sum_{j=1}^{N} (u | P_s w_j) g_j .
\]

so if we define the operator \( T'' \) as

\[
T''u = \sum_{j=1}^{N} (u | P_s w_j) g_j .
\]

we see that the decoupled case considered above corresponds to the case where \( T'' = 0 \).

The operator \( T_1 \), defined as

\[
T_1 u = \gamma u - T'u - T''u
\]

defines a normal boundary condition \( T_1 u = 0 \) (in the sense of Grubb [5]), hence the operator realization \( A_{T_1} \) of \( A \), with domain

\[
D(A_{T_1}) = \{ u \in H^{2m}(\Omega) | T_1 u = 0 \}
\]

is a densely defined, closed operator in \( L^2(\Omega) \). Here \( T_1 \) can be regarded as a perturbation of the trace operator \( T = \gamma - T' \) since

\[
T_1 = \gamma - T' - T'' = T - T''
\]

Let \( K_T \) be the Poisson solution operator defined by \( u = K_T \varphi \), where \( u \) is the solution of

\[
\begin{align*}
Au &= 0 \quad \text{in } \Omega \\
T u &= \varphi \quad \text{on } \Gamma
\end{align*}
\]
We use here that $A_r$ is made bijective, so $K_r$ is well defined. Observe also the estimate

\[
(5.71) \quad ||K_{r}T''u||_o = \sum_{j=1}^{N} \langle u|P_s w_j \rangle K_{r}g_j \|_o
\]

\[
\leq ||u||_o \sum_{j=1}^{N} ||P_s w_j||_o \|K_{r}g_j||_o .
\]

**Lemma 5.8**

There exists a constant $r_1 > 0$, such that for $||P_s w_j||_o < r_1$, $j = 1, 2, ..., N$ the operator

\[
(5.72) \quad 1 - K_{r}T''
\]

is a homeomorphism in $H^{2m}(\Omega)$, and, in particular, defines a bijection

\[
(5.73) \quad 1 - K_{r}T'' : D(A_{r_1}) \sim D(A_r).
\]

**Proof:**

Let $r_1 > 0$ be chosen, such that for $||P_s w_j||_o < r_1$, $j = 1, 2, ..., N$, we have $||K_{r}T''||_{L^2,L^2} \leq \frac{1}{2}$. This is possible by (5.71). Now $1 - K_{r}T''$ is a bounded operator in $L^2(\Omega)$ and is inverted by a Neumann series

\[
(5.74) \quad (1 - K_{r}T'')^{-1} = \sum_{m=0}^{\infty} (K_{r}T'')^m.
\]

converging in the operator norm in $L^2(\Omega)$.

Since $K_{r}$ has range in $H^{2m}(\Omega)$, we see that $1 - K_{r}T''$ is a homeomorphism in $H^{2m}(\Omega)$, (see also § 4, where $1 - K_{r}T$ was discussed) and, since

\[
(5.75) \quad T(1 - K_{r}T'')u = Tu - T''u = T'u,
\]

$1 - K_{r}T''$ defines a bijection from $D(A_{r_1})$ onto $D(A_r)$. $\square$
By Lemma 5.8 we have that if \( u \in D(A_{T_1}) \), then \( v = (1 - K_{\Gamma}T'')u \in D(A_{\Gamma}) \), and \( Au = Av \) in this case.

We will now study the resolvent \( R(\lambda, A_{T_1}) \) of \( A_{T_1} \), and we start out with the equation

\[
(5.76) \quad \begin{cases} 
(A - \lambda)u = f \quad \text{in } \Omega \\
T_1u = 0 \quad \text{on } \Gamma
\end{cases}
\]

Using (5.73) with \( v = (1 - K_{\Gamma}T'')u \) we get

\[
(5.77) \quad \begin{cases} 
(A - \lambda)(v + K_{\Gamma}T''u) = f \quad \text{in } \Omega \\
Tv = 0 \quad \text{on } \Gamma
\end{cases}
\]

so that

\[
(5.78) \quad \begin{cases} 
(A - \lambda)v = f + \lambda K_{\Gamma}T''u \quad \text{in } \Omega \\
Tv = 0 \quad \text{on } \Gamma
\end{cases}
\]

or

\[
(5.79) \quad \begin{cases} 
v = R(\lambda, A_{\Gamma})(f + \lambda K_{\Gamma}T''u) \quad \text{in } \Omega \\
Tv = 0 \quad \text{on } \Gamma
\end{cases}
\]

where the resolvent \( R(\lambda, A_{\Gamma}) \) of \( A_{\Gamma} \) is used.

Then

\[
(5.80) \quad v = u - K_{\Gamma}T''u = R(\lambda, A_{\Gamma})(f + \lambda K_{\Gamma}T''u)
\]

so that

\[
(5.81) \quad (1 - (K_{\Gamma}T'' + \lambda R(\lambda, A_{\Gamma})K_{\Gamma}T''))u = R(\lambda, A_{\Gamma})f
\]

For \( \varepsilon > 0 \), \( 0 < \theta < \frac{\pi}{2} \) we define the sector

\[
\mathcal{W}_{\lambda_K, \varepsilon, \theta} = \{ z \in \mathbb{C} | z = (\lambda_K - \varepsilon) + re^{i\omega}, \ r \geq 0, \ \frac{\pi}{2} - \theta < \omega < \frac{3\pi}{2} + \theta \}
\]

and from the estimates (5.49) and (5.71) we see that for any \( \varepsilon > 0 \).
any \( \theta \in ]0, \frac{\pi}{2}\) \], there exists a constant \( r > 0 \) such that for \( w_j \in L^2(\Omega) \), satisfying \( \| P_s w_j \|_0 < r \leq r_1 \), \( j = 1, 2, \ldots, N \), we have

\[
(5.82) \quad \| K_{T'} + \lambda R(\lambda, A_T) K_{T'} \|_{L^2, L^2} \leq \frac{1}{2}
\]

for all \( \lambda \in W_{\lambda K}, \epsilon, \theta \).

With the \( w_j, j = 1, 2, \ldots, N \), chosen this way, the resolvent \( R(\lambda, A_T) \) of \( A_T \) is a well defined, bounded operator in \( L^2(\Omega) \), given by (see (5.73))

\[
(5.83) \quad R(\lambda, A_T) = (1 - (K_{T'} + \lambda R(\lambda, A_T) K_{T'}))^m R(\lambda, A_T)
\]

\[
= \sum_{m=0}^{\infty} (K_{T'} + \lambda R(\lambda, A_T) K_{T'})^m R(\lambda, A_T),
\]

satisfying the estimate

\[
(5.84) \quad \| R(\lambda, A_T) \|_{L^2, L^2} \leq \frac{c_1}{|\lambda - \lambda_K|},
\]

for all \( \lambda \in W_{\lambda K}, \epsilon, \theta \), where \( c_1 > 0 \) is a constant, depending only on \( \epsilon \) and \( \theta \).

We have now

**THEOREM 5.9**

Let \( \epsilon > 0 \) be given, and assume that \( P_s w_j \) and \( g_j, j = 1, 2, \ldots, N \) satisfy the hypothesis of Theorem 5.5.

Then there exists a constant \( r > 0 \), such that for arbitrary choises of \( P_s w_j \) with \( \| P_s w_j \|_0 < r \), the finite dimensional Dirichlet boundary feedback condition

\[
(5.85) \quad \gamma u = T'u + T''u = \sum_{j=1}^{N} (u|w_j)g_j
\]
defines a realization \( A_{T_1} \) of \( A \), with domain

\[
D(A_{T_1}) = \{ u \in H^{2m}(\Omega) | T_i u = \gamma u - T'u - T''u = 0 \}
\]

that is the infinitesimal generator of an analytic semigroup \( e^{-A_{T_1}^t} \), \( t \geq 0 \), on \( L^2(\Omega) \), giving the solution to the Dirichlet boundary feedback control system

\[
\begin{align*}
\partial_t u + Au &= 0 \quad \text{in } \Omega, \text{ for } t > 0 \\
\nu u &= \sum_{j=1}^N (u | w_j) g_j \quad \text{on } \Gamma, \text{ for } t > 0 \\
u u &= u_0 \quad \text{in } \Omega, \text{ at } t = 0.
\end{align*}
\]

as

\[
u u(t, x) = e^{-A_{T_1}^t} u_0(x), \quad t \geq 0, \ x \in \Omega, \ u_0 \in L^2(\Omega).
\]

The solution (5.88) satisfies

\[
\|u(t, \cdot)\|_0 \leq M' e^{-(\lambda_K^- \epsilon)t} \|u_0\|_0, \quad t \geq 0
\]

where \( \lambda_K^- \) is the first positive Dirichlet eigenvalue of \( A \), and \( M' \) is a constant, depending only on \( \epsilon \) and \( \theta : \theta \) defining the sector of analyticity \( W_{\lambda_K^- \epsilon} \).

**Remark 5.10**

According to the proof of Lemma 5.7, we have a factorization

\[
A_{T_1} = A_T (1 - K_T T''')\]

We will now use the factorization (5.90) in the investigation of the hyperbolic problem for \( A_{T_1} \).

The boundary value problem

\[
\partial^2_t u + A_{T_1} u = 0, \ u \in D(A_{T_1})
\]
transforms by (5.90) into

\[(5.92) \quad (1 - K_T T'')^{-1} \partial_t^2 v + A_T v = 0, \quad v \in D(A_T)\]

where \( v = (1 - K_T T'') u \).

Acting with \((1 - K_T T'')\) from the left in (5.92) we find

\[(5.93) \quad \partial_t^2 v + (1 - K_T T'') A_T v = 0, \quad v \in D(A_T)\]

If the \( w_j \) are chosen such that \( P_s w_j \in D(A_T^*) \), \( j = 1, 2, \ldots, N \), then for \( v \in D(A_T) \):

\[
\|K_T T'' A_T v\|_0 = \| \sum_{j=1}^{N} (A_T v | P_s w_j) K_T g_j \|_0 \\
= \| \sum_{j=1}^{N} (v | A_T^* P_s w_j) K_T g_j \|_0 \\
\leq \|v\|_0 \sum_{j=1}^{N} \|A_T^* P_s w_j\|_0 \|K_T g_j\|_0
\]

i.e. \( K_T T'' A_T \) acts like a bounded operator on \( D(A_T) \), for \( P_s w_j \in D(A_T^*) \), \( j = 1, 2, \ldots, N \).

Therefore, (5.93) and with it (5.91) is nothing but a bounded perturbation of the equation

\[
\begin{cases}
\partial_t^2 v + Av = 0 & \text{in } \Omega, \text{ for } t \in \mathbb{R} \\
Tv = 0 & \text{on } \Gamma, \text{ for } t \in \mathbb{R} \\
v = v_0 & \text{in } \Omega, \text{ at } t = 0 \\
\partial_t v = v_1 & \text{in } \Omega, \text{ at } t = 0
\end{cases}
\]

(5.94)

treated in Theorem 5.5, in this case. Since the spectrum of \( A_T \) is assumed to be contained in \( \mathbb{R}^+ \), we find from standard bounded perturba-
tion theory (see e.g. Sova [12] and Fattorini [3]) that the operators

\begin{align}
(5.95) \quad & \cos((A_T - K_T T' A_T)t) \\
(5.96) \quad & \sin((A_T - K_T T' A_T)t)
\end{align}

are well defined, bounded operators in $L^2(\Omega)$ for $P_j \in \text{D}(A_T^*), \ j = 1, 2, \ldots, N,$ and $t \in \mathbb{R}$. From Theorem 1.6.11 in Grubb [5] we find that

\begin{equation}
(5.97) \quad \text{D}(A_T^*) = \{ u \in H^{2m}(\Omega) \mid \text{i}^X s^{*1} \gamma u = 0 \}
\end{equation}

where $\text{i}^X$ is the "reflection" of the index set, replacing $\{k\}$ by $\{2m - k - 1\}_{0 \leq k \leq m}$, and $s^{*1}$ is the $m \times m$ matrix differential operator, appearing in Greens formula ((A.4) in the appendix) for $A$, it is invertible since $A$ is elliptic. We see from ex. 1.6.12. in Grubb [5] that the action of the realization $A_T^*$ is of the form $A + G$, for a certain singular Green operator $G$ of finite rank. But the condition of being in $\text{D}(A_T^*)$ is then simply to be in $H^{2m}(\Omega) \cap H^m_0(\Omega)$, that is

\begin{equation}
(5.98) \quad \text{D}(A_T^*) = \text{D}(A_\gamma).
\end{equation}

We have now

**THEOREM 5.11**

Let the set $\{w_j, g_j\}_{1 \leq j \leq N}$ be chosen according to Theorem 5.9, and assume furthermore that $P_j w_j, \ j = 1, 2, \ldots, N,$ are chosen in $\text{D}(A_\gamma) = H^{2m}(\Omega) \cap H^m_0(\Omega)$.

Then the operators

\begin{align}
(5.99) \quad & C(t) = \cos((A_T - K_T T' A_T)t) \\
(5.100) \quad & S(t) = (A_T - K_T T' A_T)^{-1/2} \sin((A_T - K_T T' A_T)^{1/2}t)
\end{align}
on $L^2(\Omega)$, are well defined for $t \in \mathbb{R}$, giving the solution to the system

$$
\begin{align*}
\frac{\partial^2 v}{\partial t^2} + (1 - K_T T') A v &= 0 & \text{in } \Omega, & \text{for } t \in \mathbb{R} \\
\nu v &= \sum_{j=1}^{N} (v|P w_j) g_j & \text{on } \Gamma, & \text{for } t \in \mathbb{R} \\
v &= v_0 & \text{in } \Omega, & \text{at } t = 0 \\
\partial_t v &= v_1 & \text{in } \Omega, & \text{at } t = 0
\end{align*}
$$

(5.101)

as

$$
(5.102) \quad v(t,x) = C(t)v_0(x) + S(t)v_1(x), \quad x \in \Omega, \quad t \in \mathbb{R}, \quad v_0, v_1 \in L^2(\Omega).
$$

The solution to the original system

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + Au &= 0 & \text{in } \Omega, & \text{for } t \in \mathbb{R} \\
\nu u &= \sum_{j=1}^{N} (u|w_j) g_j & \text{on } \Gamma, & \text{for } t \in \mathbb{R} \\
u &= u_0 & \text{in } \Omega, & \text{at } t = 0 \\
\partial_t u &= u_1 & \text{in } \Omega, & \text{at } t = 0
\end{align*}
$$

(5.103)

is then by (5.90)

$$
(5.104) \quad u(t,x) = (1 - K_T T')^{-1} C(t)(1 - K_T T')u_0(x) + (1 - K_T T')^{-1} S(t)(1 - K_T T')u_1(x), \quad x \in \Omega, \quad t \in \mathbb{R}, \quad u_0, u_1 \in L^2(\Omega).
$$
§ 6. STABILIZATION BY PERTURBATIONS OF THE THIRD KIND.

We want to stabilize the systems (1.1) and (1.2) by changing both the boundary condition and the system operator. The stabilized systems will then take the form

\[
\begin{align*}
\partial_t u + Au + Gu &= 0 \quad \text{in } \Omega, \text{ for } t > 0 \\
\gamma u &= T' u \quad \text{on } \Gamma, \text{ for } t > 0 \\
u &= u_0 \quad \text{in } \Omega, \text{ at } t = 0
\end{align*}
\]

(6.1)

and

\[
\begin{align*}
\partial^2_t u + Au + Gu &= 0 \quad \text{in } \Omega, \text{ for } t \in \mathbb{R} \\
\gamma u &= T' u \quad \text{on } \Gamma, \text{ for } t \in \mathbb{R} \\
u &= u_0 \quad \text{in } \Omega, \text{ at } t = 0 \\
\partial_t u &= u_1 \quad \text{in } \Omega, \text{ at } t = 0
\end{align*}
\]

(6.2)

and we say that the systems (6.1) and (6.2) are associated with the realization \( B \), where

\[
B = (A + G)_T
\]

(6.3)

\[
D(B) = \{ u \in H^{2\alpha}(\Omega) \mid Tu = 0 \}
\]

(6.4)

\[
T = \gamma - T'.
\]

(6.5)

We will now determine operators \( G \) and \( T' \), such that (6.1) and (6.2) are stable systems of feedback type. Moreover, we will make \( B \) a selfadjoint generator of an analytic contraction semigroup.

As in the preceding paragraph, let \( \{ \varphi_j \}_{j \geq 1} \) be an orthonormalized set of eigenfunctions for \( A_\gamma \), enumerated according to the ordering of the eigenvalues (5.5), but define now the boundary feedback operator \( T' \) as

\[
T'u = \sum_{j=1}^{K-1} (u|\varphi_j)h_j
\]

(6.6)

where \( K - 1 \) is the number of negative eigenvalues of \( A_\gamma \), repeated according to multiplicity, and
\[(6.7) \quad h_j \in C^\infty(\Gamma)^m, \quad j = 1, 2, \ldots, K - 1\]

are chosen linearly independent, with no other conditions on them. Moreover, define the operator \( G \) as

\[(6.8) \quad G = K'u + G'\]

where \( u \) is the Neumann trace operator \((1.6)\) and

\[
\begin{align*}
K' &= -T^\tau_{\Omega} \mathfrak{a}^\Omega  \\
G' &= -T^\tau_{\Omega} (\mathcal{G} - c)\gamma
\end{align*}
\]

Here \( c \) is a positive constant to be determined later, \( \mathfrak{a}^\Omega \) is the upper right corner in the coefficient matrix \((A,3)\) in Greens formula (see the appendix), whereas \( \mathcal{G} \) is the coefficient matrix appearing in the boundary term in the "halfways" Greens formula for a convenient sesquilinear form \( a(u,v) \) associated with \( A \) (see \((A,6)\)). Here we will point out that the following construction is not very "economic", as we typically use a large number of feedback terms, but the construction clarifies to a great extent the interaction System operator equation ⇔ boundary equation.

Let \( a(u,v) \) be chosen such that it is \( H_0^m(\Omega) \)-coercive, then it is well known how \( A_{\gamma} \) is the variational operator associated with the triple \((a, H_0^m(\Omega), L^2(\Omega))\) (see e.g. Grubb [5], § 1.7). Let us define the sesquilinear form \( a_i(u,v) \) on \( H^m(\Omega) \) by

\[(6.10) \quad a_i(u,v) = a(u,v) + c(T'u|\gamma v)_\Gamma .\]

Let \( U \) be the space

\[(6.11) \quad U = \{ u \in H^m(\Omega) \mid \gamma u = T'u \} .\]

and observe that \( U \) is dense in \( L^2(\Omega) \), since \( T = \gamma - T' \) defines a normal boundary condition \( Tu = 0 \).
Let $B_1$ be the operator associated with the triple $(a_1, U, L^2(\Omega))$, defined as follows

\begin{equation}
\begin{aligned}
D(B_1) &= \{ u \in U | \exists f \in L^2(\Omega) \text{ so that } a_1(u, v) = (f | v) \text{ for all } v \in U \} \\
B_1 u &= f
\end{aligned}
\end{equation}

(6.12)

We will show that $B = B_1$, where $B$ is the realization defined in (6.3) - (6.5).

**Lemma 6.1**

(6.13) $B \subseteq B_1$

**Proof:**

For $u \in D(B), v \in U$ we have (see appendix)

\[ a_1(u, v) = a(u, v) + c(T' u | \gamma v)_\Gamma \]

\[ = (Au | v) - (a^{01} u + \gamma u | \gamma v)_\Gamma + c(T^* u | \gamma v)_\Gamma \]

\[ = (Au | v) - (a^{01} u + \gamma u | T' v)_\Gamma + c(\gamma u | T' v)_\Gamma \]

\[ = (Au - T^* (\gamma - c) \gamma u - T^* a^{01} u | v) \]

\[ = ((A + C) u | v) . \]

This shows that $u \in D(B_1)$ with $B_1 u = (A + C) u$, so it follows that $D(B) \subseteq D(B_1)$ with $B_1 u = Bu$ there.

For $\ell \in \mathbb{N}$ we define the subspace $W_\ell$ of $H_0^m(\Omega)$ by

\begin{equation}
W_\ell = \text{span}^m \{ \varphi_j \}_{j \geq \ell} \quad (H^m(\Omega) - \text{closure})
\end{equation}

(6.14)

**Lemma 6.2**

There exists a linearly independent set of functions $\{v_j\}_{1 \leq j < K}$ in $U \setminus W_K$, such that

\begin{equation}
U = \text{span} \{ v_j \}_{1 \leq j < K} \oplus W_K
\end{equation}

(6.15)
Proof
According to Grubb [5], lemma 1.6.8 (see our § 3) we have

(6.16) \[ U = \Lambda^{-1} H^m_0(\Omega) \]

where \( \Lambda \) and \( \Lambda^{-1} \) are bounded operators in \( H^s(\Omega) \), for all \( s \geq 0 \).

Choose a linearly independent set \( \{z_j\}_{1 \leq j < K} \) in \( H^m_0(\Omega) \) such that the matrix

(6.17) \[ C = (z_i^1 (\Lambda^{-1})^i_j \varphi_j^k)_{i,j} \]

is regular, and define

(6.18) \[ v_j = \Lambda^{-1} z_j, \quad j = 1, 2, \ldots, K-1; \]

this is clearly a linearly independent set in \( U \).

Moreover, for \( 1 \leq j < K \)

(6.19) \[ \gamma v_j = \gamma \Lambda^{-1} z_j = \sum_{k=1}^{K-1} (\Lambda^{-1} z_j | \varphi_k) h_k \]

\[ = \sum_{k=1}^{K-1} (z_j^1 (\Lambda^{-1})^i_j \varphi_j^k) h_k \neq 0 \]

since \( C \) is regular, so none of the \( v_j \) lie in \( H^m_0(\Omega) \).

Since

(6.20) \[ \gamma \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{K-1} \end{bmatrix} = C \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{K-1} \end{bmatrix} \]
the set \( \{\gamma v_j\}_{1 \leq j < K} \) is linearly independent, so \( \gamma(\sum_{j=1}^{K-1} \alpha_j v_j) = 0 \) implies that \( \alpha_j = 0 \) for \( j = 1, 2, \ldots, K-1 \).

Hence \( \text{span}\{v_j\}_{1 \leq j < K} \cap W_K = \{0\} \).

Since \( W_K \subseteq U \) and \( \text{span}\{v_j\}_{1 \leq j < K} \subseteq U \) the inclusion

\[
(6.21) \quad \text{span}\{v_j\}_{1 \leq j < K} + W_K \subseteq U
\]

is evident.

To show the inclusion the other way, let \( u \in U \) and invert (6.20) to get

\[
(6.22) \quad \gamma u = \sum_{j=1}^{K-1} (u|\varphi_j)h_j = \sum_{j=1}^{K-1} \left( (u|\varphi_j) \sum_{k=1}^{K-1} \alpha_{jk} \gamma v_k \right), \quad \alpha_{jk} \in \mathbb{C}.
\]

The last term equals \( \sum_{k=1}^{K-1} \beta_k \gamma v_k \), where

\[
(6.23) \quad \beta_k = \sum_{j=1}^{K-1} (u|\varphi_j) \alpha_{jk}, \quad k = 1, 2, \ldots, K-1.
\]

If we define

\[
(6.24) \quad w = u - \sum_{j=1}^{K-1} \beta_j v_j
\]

then

\[
(6.25) \quad \gamma w = \gamma u - \sum_{j=1}^{K-1} \beta_j \gamma v_j = 0.
\]
hence

\[ (6.26) \quad 0 = \gamma w = \sum_{j=1}^{K-1} (w | \varphi_j) h_j. \]

But the set \{h_j\}_{1 \leq j < K} is linearly independent, so (6.26) implies that

\[ (6.27) \quad (w | \varphi_j) = 0, \quad j = 1, 2, \ldots, K-1. \]

Then, from (6.24)

\[ (6.28) \quad u = \sum_{j=1}^{K-1} \beta_j v_j + w \in \text{span}\{v_j\}_{1 \leq j < K}^\perp + \overline{W}_K^o. \]

Since \overline{W}_K^o (i.e. the \(L^2(\Omega)\)-closure) has a \(L^2\)-complement \((\overline{W}_K^o)^{\perp}\) of dimension \(K-1\), we can assume that the \(v_j\) are chosen outside \(\overline{W}_K^o\). Then we have a decomposition of \(u \in U\) as

\[ (6.29) \quad u = v + w \]

where \(v \in \text{span}\{v_j\}_{1 \leq j < K}\), \(v_j \in (\overline{W}_K^o)\) \(j = 1, 2, \ldots, K-1,\) and \(w \in W_K^o\), and the projections

\[ (6.30) \quad \begin{cases} u \to v \\ u \to w \end{cases} \]

are then continuous in \(L^2(\Omega)\) so we have an inequality

\[ (6.31) \quad ||v||_o + ||w||_o \leq C||u||_o \]

with a constant \(C > 0\).

**Lemma 6.3**

\(a(u,v)\) is \(m\) - coercive on \(U\)
Proof:
Let \( u \in U \) and write \( u = v + w \) as above. Then
\[
a(u,u) = a(w,w) + 2\text{Re} \ a(v,w) + a(v,v) .
\]

For any \( \epsilon > 0 \) we can find constants \( C_1, C_2 > 0 \), such that
\[
2|a(v,w)| \leq 2C_1||w||_m||v||_m
\]
\[
\leq \epsilon^2||w||_m^2 + \frac{C_1}{\epsilon^2}||v||_m^2
\]
\[
\leq \epsilon^2||w||_m^2 + \frac{C_2}{\epsilon^2}||u||_0^2
\]
because of (6.31) and the fact that the norms are equivalent on a finite dimensional space.

Since \( a(u,v) \) is \( H^m_0(\Omega) \)-coercive, again using (6.31) we find
\[
a(w,w) \geq C_3||w||_m^2 - C_4||w||_0^2
\]
\[
\geq C_5||w||_m^2 - C_6||u||_0^2
\]
with constants \( C_3, C_4, C_5 > 0 \).

Moreover, since
\[
a(v,v) \leq C_7||v||_m^2 - C_7||v||_0^2 \leq C_8||u||_0^2
\]
with \( C_6, C_7, C_8 > 0 \), we have that
\[
a(u,u) \geq (C_3 - \epsilon^2)||w||_m^2 - \frac{C_2}{\epsilon^2} + C_6 + C_8)||u||_0^2
\]
Since the norms of \( v \) are equivalent, we have positive constants \( C' \) and \( C'' \), such that
(6.36) \[ C' \|v\|_0 \leq \|v\|_m \leq C'' \|v\|_0 \]

so for all \( \varepsilon' > 0 \):

(6.37) \[
|w|_m^2 \geq |u|_m^2 - 2|v|_m |u|_m + |v|_m^2 \\
\geq |u|_m^2 + C' \|v\|_0^2 - \left( \frac{1}{\varepsilon''^2} |v|_m^2 + \varepsilon' \|u\|_m^2 \right) \\
\geq (1 - \varepsilon''^2) |u|_m^2 + (C' - \frac{1}{\varepsilon''^2} C'') \|v\|_0^2 \\
\geq (1 - \varepsilon''^2) |u|_m^2 - (C' - \frac{1}{\varepsilon''^2} C'') C |u|_0^2
\]

Choosing \( 0 < \varepsilon'' < C_0 \) and \( 0 < \varepsilon' < 1 \) and inserting (6.37) in (6.35), we get

(6.38) \[ a(u, u) \geq C_0 |u|_m^2 - k |u|_0^2 \]

with constants \( C_0 \) and \( k \), \( C_0 \) being positive. \( \Box \)

**Lemma 6.4**

\[ B_1 = B_1^* \]

**Proof**

Since for \( u \in U \)

(6.39) \[ a_1(u, u) = a(u, u) + c(Tu|u|u)_T = a(u, u) + c(Tu|Tu)_T \geq a(u, u) \]

\( a_1 \) is \( m \)-coercive on \( U \). Then \( B_1 \) is the **variational** operator associated with the triple \( (a_1, U, L^2(\Omega)) \), and since \( a_1(u, v) \) is symmetric on \( U \), \( B_1 = B_1^* \). (See e.g. Grubb [5], § 1.7) \( \Box \)

Now we notice that the boundary value problem

(6.40) \[
\begin{cases}
    Au + Gu = f, & f \in L^2(\Omega) \\
    Tu = 0
\end{cases}
\]
is elliptic, since the Dirichlet problem for $A$

$$
(6.41) \begin{cases}
    Au = f, \ f \in L^2(\Omega) \\
    \gamma u = 0
\end{cases}
$$

is elliptic and $G$ has the form (6.8) - (6.9) where $T^*\tilde{\kappa}$ and $T'$ obviously are integral operators with $C^\infty$-kernels, hence of order $-\infty$. ((6.40) and (6.41) have the same principal symbols.) We can then use Theorem 1.6.11 and ex. 1.6.12 from Grubb [5] to show that $B = B^\star$. It is shown there that when

$$
(6.42) \quad B = (A + C)_T
$$

is an elliptic realization of a formally selfadjoint operator $A$, with

$$
(6.43) \quad \begin{cases}
    G = K'u + G' \\
    T = \gamma - T'
\end{cases}
$$

then $B = B^\star$ if

$$
(6.44) \quad \begin{cases}
    T' = -(a^{01}\tilde{\kappa})^{-1} K^\star \tilde{\kappa} \\
    G' = G^\star + T^\star a^{00}\gamma
\end{cases}
$$

Using that $K' = -T^\star a^{01} \tilde{\kappa}$ (see (6.9)) and that $a^{00} = \gamma - \gamma^\star$ (see appendix. Lemma (A.1)) we compute

$$
(6.45) \quad -(a^{01}\tilde{\kappa})^{-1} K^\star \tilde{\kappa} = (a^{01}\tilde{\kappa})^{-1} a^{01}\tilde{\kappa} T' = T'
$$

and

$$
(6.46) \quad G^\star + T^\star a^{00}\gamma = -T^\star (\gamma^\star - c)\gamma + T^\star (\gamma^\star - \gamma)\gamma
$$

$$
= -T^\star (\gamma - c)\gamma
$$

$$
= G',
$$

hence $B = B^\star$, according to (6.44).
We have now

**THEOREM 6.5**

(6.47) \[ B = B_1 \]

**Proof**

\( B \subseteq B_1 \) implies that \( B_1^\# \subseteq B^\# \). But \( B = B^\# \) and \( B_1 = B_1^\# \). \( \square \)

We have shown that the realization

(6.48) \[ B = (A + C)_T \]

is the variational operator associated with the triple \((a_1, U, L^2(\Omega))\).

**THEOREM 6.6**

Given \( \zeta < 1 \), there exists a constant \( c > 0 \) such that the sesquilinear form

(6.49) \[ a_1(u,v) = a(u,v) + c(Tu|Tv)_\Gamma \]

satisfies, for all \( u \in U \):

(6.50) \[ a_1(u,u) \geq \zeta \lambda_K ||u||_0^2 . \]

Here \( \lambda_K \) is the first positive Dirichlet eigenvalue of \( A \).

**Proof**

We write \( u \in U \) as \( u = v + w \) as in (6.29).

Then

\[
\frac{a_1(u,u)}{||u||_0^2} = \frac{a(v+w,v+w) + c(T(v+w)|v+w)_\Gamma}{(v+w|v+w)} = \frac{a(v+w,v+w) + c(T(v+w)|T(v+w))_\Gamma}{(v+w|v+w)} = \frac{a(w,w) + 2Re(a(v,w) + a(v,v) + c||Tv||_0^2}{||w||_0^2 + 2Re(v|w) + ||v||_0^2}
\]
Now since

\begin{equation}
\tag{6.51}
a(w,w) \geq C_1 \|w\|_m^2 - C_2 \|w\|_0^2, \quad C_1, C_2 > 0.
\end{equation}

we have, for all \( \epsilon > 0 \):

\begin{equation}
\tag{6.52}
2 |a(v,w)| \leq \epsilon^2 \|w\|_m^2 + \frac{C_3}{\epsilon^2} \|v\|_m^2
\end{equation}

\begin{align*}
&\leq \epsilon^2 \|w\|_m^2 + \frac{C_4}{\epsilon^2} \|v\|_0^2 \\
&\leq \epsilon^2 \left( \frac{1}{C_1} a(w,w) + \frac{C_2}{C_1} \|w\|_0^2 \right) + \frac{C_4}{\epsilon^2} \|v\|_0^2 \\
&\leq \epsilon^2 \left( \frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1} \right) a(w,w) + \frac{C_4}{\epsilon^2} \|v\|_0^2
\end{align*}

with constants \( C_0, C_4 > 0 \).

Moreover, for all \( \epsilon' > 0 \):

\begin{equation}
\tag{6.53}
2 |(v\quad w)| \leq 2 \|v\|_0 \|w\|_0 \leq \epsilon'^2 \|w\|_0^2 + \frac{1}{\epsilon'^2} \|v\|_0^2
\end{equation}

so we find that

\begin{equation}
\tag{6.54}
\frac{a_1(u,u)}{\|u\|_0^2} \geq \left( 1 - \epsilon^2 \left( \frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1} \right) \right) \frac{a(w,w) - \left( \frac{C_4}{\epsilon^2} + C_6 \right) \|v\|_0^2 + c \|Tv\|_0^2}{(1 + \epsilon'^2) \|w\|_0^2 + (1 + \frac{1}{\epsilon'^2}) \|v\|_0^2}
\end{equation}

where we also used that \( |a(v,v)| \leq C_5 \|v\|_m^2 \leq C_5 \|v\|_0^2 \), \( C_5, C_6 > 0 \).

Then

\begin{equation}
\tag{6.55}
\frac{a_1(u,u)}{\|u\|_0^2} \geq \left( 1 - \epsilon^2 \left( \frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1} \right) \right) \frac{\lambda_K \|w\|_0^2 - \left( \frac{C_4}{\epsilon^2} + C_6 \right) \|v\|_0^2 + c \|Tv\|_0^2}{(1 + \epsilon^2) \|w\|_0^2 + (1 + \frac{1}{\epsilon'^2}) \|v\|_0^2}
\end{equation}
T is injective from \( \text{span}\{v_j\}_{1 \leq j < K} \) to \( \text{span}\{h_j\}_{1 \leq j < K} \), so we have that

\[
||Tv||_0^2 \geq C_7 ||v||_0^2 .
\]

with a positive constant \( C_7 \).

Hence

\[
a_1(u,u) \geq \frac{a\lambda_K ||w||_0^2 + \beta ||v||_0^2}{\mu ||w||_0^2 + \theta ||v||_0^2}
\]

where

\[
\alpha = 1 - \varepsilon^2 \left( \frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1} \right) , \beta = cC_7 - \frac{C_4}{\varepsilon^2} - C_6
\]

\[
\mu = 1 + \varepsilon^2 , \theta = 1 + \frac{1}{\varepsilon^2}
\]

Considering the function

\[
f(s) = \frac{a\lambda_K s + \beta}{\mu s + \theta} , s > 0 .
\]

which is decreasing for \( \lambda_K a\theta - \mu \beta < 0 \), we see that

\[
a_1(u,u) \geq \frac{\alpha}{\mu} \lambda_K
\]

if

\[
\beta > \frac{a\theta}{\mu} \lambda_K ,
\]

so if

\[
c > \frac{(1 - \varepsilon^2 \left( \frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1} \right) \lambda_K (1 + \frac{1}{\varepsilon^2})}{C_7(1 + \varepsilon^2)} + \frac{C_4}{C_7 \varepsilon^2} + \frac{C_6}{C_7}
\]
then

$\quad a_1(u,u) \geq \frac{1 - \epsilon^2 \left( \frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1} \right)}{1 + \epsilon^2} \lambda_K \|u\|^2_0.$

With $\epsilon$ and $\epsilon'$ chosen such that

$\frac{1 - \epsilon^2 \left( \frac{1}{C_1} + \frac{1}{\lambda_K} \cdot \frac{C_2}{C_1} \right)}{1 + \epsilon^2} = \zeta$ and $c$ chosen such that (6.61) holds, we see that

$\quad a_1(u,u) \geq \zeta \lambda_K \|u\|^2_0$

as claimed. $\square$

We have now

**THEOREM 6.7**

Given any $\zeta < 1$, there exists a constant $c > 0$ such that the operator realization

$\quad B = (A + C)_T$ (see (6.3) - (6.5))

has its spectrum in the halfline $[0, \infty)$, and is the infinitesimal generator of an analytic semigroup $e^{-Bt}$, $t \geq 0$, on $L^2(\Omega)$, giving the solutions to the parabolic system

$\left\{ \begin{array}{l}
\partial_t u + Au + Gu = 0 \quad \text{in } \Omega, \text{ for } t > 0 \\
\gamma u = T'u \quad \text{on } \Gamma, \text{ for } t > 0 \\
u = u_0 \quad \text{in } \Omega, \text{ at } t = 0
\end{array} \right.$

as

$\quad u(t,x) = e^{-Bt} u_0(x), \quad x \in \Omega, \quad t \geq 0, \quad u_0 \in L^2(\Omega).$

The solution satisfies

$\quad \|u(t, \cdot)\|_0 \leq e^{\zeta \lambda_K t} \|u_0\|_0, \quad t \geq 0.$
Moreover the operators

\begin{align*}
(6.69) \quad \begin{cases}
    C(t) = \cos (B^{1/2} t) \\
    S(t) = B^{-1/2} \sin (B^{1/2} t)
\end{cases}
\end{align*}

are well defined for \( t \in \mathbb{R} \), giving the solution to the hyperbolic problem

\begin{align*}
(6.70) \quad \begin{cases}
    \frac{\partial^2 u}{\partial t^2} + Au + Cu = 0 \quad \text{in } \Omega, \text{ for } t \in \mathbb{R} \\
    \nu u = T'u \quad \text{on } \Gamma, \text{ for } t \in \mathbb{R} \\
    u = u_0 \quad \text{in } \Omega, \text{ at } t = 0 \\
    \partial_t u = u_1 \quad \text{in } \Omega, \text{ at } t = 0
\end{cases}
\end{align*}

as

\begin{align*}
(6.71) \quad u(t,x) = C(t)u_0(x) + S(t)u_1(x), \quad x \in \Omega, \quad t \in \mathbb{R}.
\end{align*}

when \( u_0, u_1 \in L^2(\Omega) \). \( \square \)

\textbf{Remark 6.8}

In the light of Proposition 1.7.11 in Grubb [5], it is of interest to notice that the realization constructed above is weakly semibounded, because we compensate for the non-localness in the boundary condition by non-local terms in the system operator equation, satisfying

\begin{align*}
(6.72) \quad K^M = -w_0^M \Gamma'.
\end{align*}

Moreover, we can notice that \( B \) is selfadjoint and is \( m \)-bounded, that is

\begin{align*}
(6.73) \quad |(Bu|v)| \leq C_0 ||u||_m ||v||_m \quad \text{for } u, v \in D(B). \quad \square
\end{align*}

\textbf{Example 6.9}

Let us calculate the operator \( G \) in the case where \( A = -\Delta \).

Since

\begin{align*}
(6.74) \quad T'u = \sum_{j=1}^{K-1} (u|\varphi_j)h_j.
\end{align*}
we have that

\[(6.75)\quad \mathbb{T}^* \psi = \sum_{j=1}^{K-1} (\psi | h_j) \mathbb{T} \varphi_j .\]

hence

\[(6.76)\quad \mathbb{G}u = -\mathbb{T}^* \mathbb{F}^{01} uu - \mathbb{T}^*(\nabla - c) \tau u \]

\[= - \sum_{j=1}^{K-1} (\mathbb{F}^{01} uu - (\nabla - c) \tau u | h_j) \mathbb{T} \varphi_j .\]

Now the terms in Greens formula (see appendix (A.4)) are particularly simple since

\[(6.77)\quad \mathbb{F} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,\]

\[(6.78)\quad (-\Delta u | v) - (u | -\Delta v) = i(uu | \tau v) \mathbb{T} + i(\tau u | uv) \mathbb{T} .\]

Then (6.76) reduces to

\[(6.79)\quad \mathbb{G}u = - \sum_{j=1}^{K-1} (i uu + c \tau u | h_j) \mathbb{T} \varphi_j .\]

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APPENDIX: GREENS FORMULAS.

For the formally selfadjoint operator A (1.3) we have

\[ (Au|v) - (u|Av) = (d\rho u|\rho v)_\Gamma. \]

(see 1.7). Here \( d \) is a skew-triangular \( 2m \times 2m \) - matrix of differential operators over the boundary \( \Gamma \), of the form

\[ d = d^0 + d^1 = \begin{bmatrix} s_1^0 & \cdots & s_{2m-1}^0 & s_{2m}^0 \\ s_2^0 & \cdots & s_{2m}^0 & 0 \\ \vdots & \ddots & \vdots \\ s_{2m}^0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} \text{lower order} \\ \text{order} \end{bmatrix} \]

where \( s_k^0 \) is of order \( 2m - k \) (See Grubb [5], § 1.3).

We usually write \( d \) in \( m \times m \) - blocks as

\[ d = \begin{bmatrix} d^{00} & d^{01} \\ d^{10} & 0 \end{bmatrix} \]

and we have the following version of Greens formula

\[ (Au|v) - (u|Av) = (d^{01}u|u + d^{00}u|\gamma v)_\Gamma + (d^{10}u|\gamma v)_\Gamma \]

for \( u, v \in H^{2m}(\Omega) \).

Now consider a symmetric sesquilinear form

\[ a(u,v) = \sum_{|\alpha|,|\beta| \leq m} (a_{\alpha\beta} \partial^\beta u|\partial^\alpha v) \]

associated with A. For such a form we have a "halfways" Greens formula, for \( u \in H^{2m}(\Omega) \) and \( v \in H^m(\Omega) \):

\[ (Au|v) - a(u,v) = (d^{01}u|u + \gamma u|\gamma v)_\Gamma \]

where the operator \( \gamma \) is of the same type as \( d^{00} \) in (A.3). Since \( d^{01\ast} = d^{10} \) when A is formally selfadjoint, we have
LEMMA A.1.

(A.7) \(.s^{00} = \gamma - \gamma^*\)

Proof.

For \(u, v \in H^2_m(\Omega)\) we have:

\[
(Au|v) = a(u,v) + (s^{01} uu + \gamma^* \gamma v|\gamma v)_\Gamma
\]

\[
(u|Av) = (Av|u) = a(u,v) + (s^{01} uu + \gamma^* \gamma v|\gamma u)_\Gamma
\]

\[
= a(u,v) + (s^{01*} \gamma u|\gamma v)_\Gamma + (\gamma^* \gamma u|\gamma v)_\Gamma
\]

so that

(A.8) \((Au|v) - (u|Av) = (s^{01} uu + (\gamma - \gamma^*) \gamma \gamma v|\gamma v)_\Gamma + (s^{10} \gamma u|\gamma v)_\Gamma\)

Comparing (A.8) with (A.4) gives us (A.7).\(\square\)

Now let \(K_\gamma\) be the Poisson solution operator to the Dirichlet problem for \(A\), i.e. \(u = K_\gamma \psi\), where

(A.9) \[
\begin{cases}
A u = 0 & \text{in } \Omega \\
\gamma u = \psi & \text{on } \Gamma
\end{cases}
\]

We can then specify the action of \(K_\gamma\) the following way:

PROPOSITION A.2.

The Poisson solution operator \(K_\gamma\) to the Dirichlet problem for \(A\) satisfies

(A.10) \((K_\gamma \psi|\varphi_j) = \frac{-1}{\lambda_j} (\psi|s^{10\gamma} \psi_j)_\Gamma\), \(j \geq 1\).

Proof.

Just insert \(u = K_\gamma \psi\) and \(v = \varphi_j\) (the eigenfunctions of \(A_\gamma\)) in (A.4), and use that \(\gamma v = 0\). Moreover, \(\lambda_j = \lambda_j^*\) since \(A_\gamma\) is selfadjoint.\(\square\)
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