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On the Pommerenke–Levin–Yoccoz inequality

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Abstract. We consider a repelling or parabolic periodic point α on the boundary of a simply connected n-periodic Fatou domain Λ for a rational map $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$. First we define a rotation number for α, if there exists a periodic access to α in Λ. Secondly we give a sufficient criterion for the existence of a periodic access to α in Λ, this generalizes a theorem about the landing of external rays for polynomials, proved by Douady, and Eremenko and Levin. Thirdly let $\phi_\Lambda : \Lambda \to \mathbb{D}$ be a Riemann map of Λ and define $R_\Lambda := \phi \circ R^n \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$, then a periodic access to α from Λ corresponds to a periodic point $\alpha' \in \partial\mathbb{D}$ for R_Λ. We prove an inequality, which relates the multiplier of α to the rotation number of α and the multiplier of α'. The essential idea of the proof is to study the modulus of a non-trivial annulus in a torus. The inequality merges inequalities obtained by Pommerenke, Levin and Yoccoz.

0. *Introduction*

Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Iteration of R gives a dynamical system with orbits $\{z_n\}_{n\geq 0}$, $z_{n+1} = R(z_n)$. For a periodic point $z \in \bar{\mathbb{C}}$ the minimal $k \in \mathbb{N}$ with $R^k(z) = z$ is the (exact) period of z and the number $\lambda = (R^k)'(z)$ is the multiplier of (the orbit of) z. A point z is preperiodic if $R^l(z)$ is periodic for some $l \geq 0$ and strictly preperiodic if it is not itself periodic. The periodic point z is attracting if $|\lambda| < 1$, repelling if $|\lambda| > 1$, indifferent if $|\lambda| = 1$ and rationally indifferent or parabolic if λ is a root of unity.

The Fatou set F_R for R is the set of points $z \in \bar{\mathbb{C}}$, for which the family of iterates $\{R^n\}_{n>0}$ forms a normal family in some neighbourhood of z. The Julia set J_R for R is the complement of the Fatou set. The Fatou set is open, and we use the name Fatou domain to denote a connected component of the Fatou set. According to Sullivan's theorem on no wandering domains for rational maps [S], any Fatou domain Λ is preperiodic. Further if Λ is periodic, then it is of one of the following types: (a) immediate attracted basin for an attracting periodic point or (b) a component of the immediate attracted basin for a parabolic periodic point or (c) a Siegel disc or an Arnold–Herman ring. The types (a), (b) and (c) we call respectively hyperbolic, parabolic and rotation domains. Hyperbolic and parabolic domains can be either simply connected or infinitely connected, while rotation domains are either a disc or a ring (annulus). For a more profound account of the theory of iteration of rational maps the reader may consult Blanchard **[B]** or Lyubich **[Ly]**.

1. *Definitions and statement of theorems*

Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map and suppose Λ is a finitely connected Fatou domain for R. An access $(\alpha, [\gamma])$ in Λ (defined in §2) is periodic, if there exists $m \in \mathbb{N}$ such that γ and $R^m(\gamma)$ define the same access in Λ. If $(\alpha, [\gamma])$ is a periodic access, we say α is periodically accessible from Λ. Periodic accessibility of α from Λ clearly implies that α and Λ are periodic. Note that a rotation domain cannot contain a periodic access, because the dynamics on such a domain is biholomorphically conjugate to an irrational rotation.

Suppose $(\alpha, [\gamma])$ is a periodic access in Λ. Let α have period $k \in \mathbb{N}$ and let kq, $q \in \mathbb{N}$ be the period of $(\alpha, [\gamma])$. The equivalence class $[\gamma]$ contains an arc γ', the center curve, with $\gamma' \subset R^{kq}(\gamma')$ or $R^{kq}(\gamma') \subset \gamma'$ (see the Complement of Proposition 2.1). Henceforth we shall assume that γ is such an arc. Also one finds that α is either repelling or parabolic with multiplier a qth root of unity (Proposition 2.1).

The q curves $\gamma, R^k(\gamma), \ldots, R^{k(q-1)}(\gamma)$ have a natural cyclic order around α: let Δ be a small closed disc centered at α, such that suitable restrictions of the q curves are arcs in Δ from α to $\partial\Delta$ intersecting only at α. The landing points of the arcs on $\partial\Delta$ have a cyclic order. The cyclic order of the curves is this cyclic order. To see that the cyclic order does not depend on the disc Δ one can use, e.g., the Jordan curve theorem. Number the curves γ_0 through γ_{q-1} in the counter-clockwise direction. Since R^k is locally injective at α, it preserves their cyclic order, thus there exist $p \in \{0, \ldots, q-1\}$, $(p, q) = 1$ with

$$R^k(\gamma_j) \supseteq \gamma_{(j+p) \bmod q} \quad (\text{or } R^k(\gamma_j) \subseteq \gamma_{(j+p) \bmod q}) \quad j = 0, \ldots, q-1.$$

We say the curves perform a p/q-rotation around α under iteration by R^k, and that they or their accesses form a p/q-cycle. (See also Goldberg and Milnor [G–M]).

Definition 1. A repelling or parabolic k-periodic point α has *(combinatorial) rotation number* p/q, if there exists a periodic access $(\alpha, [\gamma])$ in a simply connected Fatou domain Λ and γ performs a p/q-rotation around α under iteration by R^k.

The rotation number is well defined, if it exists. This is one of the consequences of the following Theorem A. Remark also that there is no reasonable definition of rotation number if Λ is, say the complement of a Cantor set. Suppose the two disjoint open finitely connected sets Λ_1 and Λ_2 contains the accesses $(\alpha, [\gamma_1])$ and $(\alpha, [\gamma_2])$ respectively. We then say that $(\alpha, [\gamma_1])$ is an exterior access for Λ_2 and vice versa.

THEOREM A. *Let Λ be a finitely connected Fatou domain and let $\alpha \in \partial\Lambda$. Suppose there exist a qk-periodic access or exterior access $(\alpha, [\kappa])$ for Λ. If α is accessible from Λ then:*

(a) *Every access to α in Λ is periodic with period qk.*
(b) *The period n of Λ divides qk.*
(c) *The number of accesses $(\alpha, [\gamma])$ to α in Λ is finite.*

One should compare Theorem A to the first statement in Theorem 2 and the statement succeeding Theorem 2 in the paper of Pommerenke [Po] (see also [G–M]).

Rotation numbers arise naturally. For a parabolic period point the local dynamics is homeomorphically conjugate to the composition of a rational rotation

$$z \mapsto z \cdot \exp(i2\pi p/q), \quad q > p \in \mathbb{N}_0, \quad (p, q) = 1$$

and a map of the form $z \mapsto z \cdot (1 + z^{nq})$ (see Camacho [Ca]). For such a point the rotation number is p/q. For a repelling periodic point the existence of a rotation number expresses that the local dynamics has the same topological properties as the composition of a rational rotation and a dilation.

Let Λ be a periodic finitely connected Fatou domain and suppose α is a repelling or parabolic periodic boundary point of Λ. Is α periodically accessible from Λ? We have a partial but incomplete answer to this question. First, if Λ is a rotation domain, then as noted above, α cannot be periodically accessible. Thus if α also has a rotation number, then by Theorem A above α is not even accessible from Λ. Moreover it is still an open question, if a rotation domain can have a periodic boundary point at all? Secondly, if $\partial\Lambda$ is locally connected, in particular if J_R is locally connected, then α is always periodically accessible from Λ (see [**Po**, Theorem 2]). Our partial answer is a generalized version of the Douady landing theorem for polynomials, Corollary B.1. A. Eremenko and G. M. Levin have proved the Douady Landing Theorem for repelling periodic points of polynomials with arbitrary Julia set, using a different approach (see [**E–L**]). It follows immediately from our Theorems A and B, that the Douady landing theorem remains valid for parabolic periodic points. This result has been obtained simultaneously and independently by Milnor (see [**M**]) and the author.

THEOREM B. (Extended Douady landing theorem.) *Let α be a repelling or parabolic periodic boundary point of a simply or doubly connected Fatou domain Λ for R. The point α is periodically accessible from Λ provided α does not belong to the closure of a non-periodic connected component of $R^{-m}(\Lambda)$ for any $m > 0$.*

COROLLARY B.1. (Douady, 1987.) *Let P be a polynomial with connected Julia set and let α be a repelling or parabolic periodic point for P. Then α is the landing point of at least one and at most finitely many external rays, all of which are periodic and defines the same rotation number.*

COROLLARY B.2. *If the repelling or parabolic periodic point α belongs to the boundary of the rotation domain Λ. Then α also belongs to the boundary of a non-periodic connected component of $R^{-m}(\Lambda)$ for some $m > 0$.*

Question. Can the closure of a strictly preperiodic component of the Fatou set for a rational map contain a parabolic or repelling periodic point?

A polynomial-like mapping (see Douady and Hubbard [**D–H-3**]) of degree d is a triple (U, U', f), where $U' \subset\subset U$ are open subsets of $\mathbb{C}$, conformally equivalent to discs and $f : U' \to U$ is a proper holomorphic mapping of degree d. The filled Julia set of f is $K_f := \{z \in U' | f^n(z) \in U' \ \forall n \geq 0\}$. Two polynomial-like mappings (U, U', f) and (V, V', g) are said to be hybrid equivalent, if there exist open neighbourhoods Ω_f and Ω_g of K_f and K_g, respectively, and a quasi-conformal homeomorphism $\Phi : \Omega_f \to \Omega_g$, such that $\Phi \circ f = g \circ \Phi$ and $\bar\partial\Phi = 0$ a.e. on K_f. We shall make use of the following theorem.

THE STRAIGHTENING THEOREM. ([**D–H-3**, Theorem 1].) *Any polynomial-like mapping (U, U', f) of degree d is hybrid equivalent to some polynomial P of degree d. Furthermore if K_f is connected, the polynomial P is unique up to affine conjugation.*

In the following we shall consider only polynomial-like mappings (U, U', f) for which K_f is connected (or equivalently K_f contains all the critical points of f). Then $U'\backslash K_f$ and $U\backslash K_f$ are annuli and the restriction $f_| : U'\backslash K_f \to U\backslash K_f$ is a covering map of degree d. The notion of rotation number naturally extends to include repelling and parabolic periodic points for f in the Julia set $J_f := \partial K_f$. Also a hybrid equivalence between polynomial-like mappings preserves repelling and parabolic periodic points and their accesses. The following is an immediate corollary of The Straightening Theorem and Corollary B.1.

COROLLARY B.3. *Let (U, U', f) be a polynomial-like mapping with K_f connected and let α be a repelling or parabolic periodic point. Then there is at least one and at most finitely many periodic accesses $(\alpha, [\gamma])$ in $U\backslash K_f$, all of which define the same rotation number. Furthermore if Φ is a hybrid equivalence between f and a polynomial P, then α and $\Phi(\alpha)$ have the same rotation number.*

Let Λ be a n-periodic, simply connected, hyperbolic or parabolic Fatou domain and let $\phi : \Lambda \to \mathbb{D}$ be a Riemann map. Then the map $R_\Lambda = \phi \circ R^n \circ \phi^{-1}$ is the restriction to $\mathbb{D}$ of a finite Blaschke product. Furthermore ϕ induces a 1–1 correspondence between the periodic accesses in Λ and the periodic points for R_Λ in $\mathbb{S}^1$ (Complement of Proposition 2.1). Let $(\alpha, [\gamma])$ be a periodic access in Λ and let α' be the corresponding periodic point for R_Λ. The multiplier $\lambda' > 1$ of α' depends neither on the choice of ϕ, nor on the choice of representative of the p/q-cycle of the access $(\alpha, [\gamma])$. We call it the *conjugate multiplier* corresponding to $(\alpha, [\gamma])$ or to the p/q-cycle of $(\alpha, [\gamma])$.

For $\alpha \in \mathbb{C}$ and $W \subset \mathbb{C}$ a Borel subset let Area(W, ρ_α) denote the area of W with respect to the conformal metric $\rho_\alpha := |dz|/|z - \alpha|$. Let U be a Borel subset of $\mathbb{C}$. The following limit if it exists, is called the logarithmic density of U at α

$$B := \lim_{\delta\to 0} \frac{\text{Area}((U \cap A(\alpha, \delta, r)), \rho_\alpha)}{\text{Area}(A(\alpha, \delta, r), \rho_\alpha)}$$

where $r > 0$ and $A(\alpha, \delta, r) = \{z \in \mathbb{C} | \delta < |z - \alpha| < r\}$. The lograrithmic density of U, whenever it exists, is a conformal invariant, which satisfies $0 \leq B \leq 1$ and which does not depend on r. Levin has proved in [L], that any domain U with $U \subseteq \lambda U$ has a logarithmic density at 0. Let α be a repelling k-periodic point and suppose $(\alpha, [\gamma])$ belongs to a p/q-cycle of periodic accesses. Choose r so small that R^k is injective on $\mathbb{D}(\alpha, r)$. Let U be the union of the q connected components of $F_R \cap \mathbb{D}(\alpha, r)$ containing the q germs of curves in the p/q-cycle of γ. The set U has a logarithmic density B at α ([L] or Proposition 4.3). We call B the *opening area* of the p/q-cycle of $(\alpha, [\gamma])$. The opening area gives a measure of the 'angular' space (opening) taken up by the p/q-cycle of $(\alpha, [\gamma])$.

THEOREM C. (The PLY-inequality for a periodic point.) *Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map and suppose α is a repelling periodic point with rotation number p/q and multiplier $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$. Let $\lambda_1, \ldots, \lambda_N$ be conjugate multipliers corresponding to distinct p/q-cycles of periodic accesses to α and let $0 \leq B \leq 1$ be the sum of the opening areas of those*

p/q-cycles. Then there exists a logarithm $L \in \mathbb{H}_+$ of λ, i.e. $\exp(L) = \lambda$ *such that:*

$$|L - (p/q) \cdot 2\pi i| \leq B \cdot \frac{2\sin\theta}{q^2 \sum_{i=1}^{N} \left(\frac{1}{\log \lambda_i}\right)},$$

where θ is the angle between $2\pi i$ and $L - (p/q)2\pi i$.

To produce his inequality Pommerenke [Po] worked with a sufficiently high iterate of R, (in fact at least R^{kq}) fixing both α and its accesses. In this way the rotation number p/q never appears with other than the trivial value 0/1. His inequality is thus really an inequality for λ^q. When we keep track of the rotation number, it corresponds to knowing which of the q roots of λ^q, λ is. Also Pommerenke did not have the logarithmic density factor B, which was found by Levin [Le]. Finally Yoccoz [Y] introduced the rotation number, but worked only with polynomials.

Theorem D and its Corollaries D.1, D.2 provides uniform estimates for multipliers, suitable for parameter space study. In fact Yoccoz was led to his inequality, Corollary D.1, by the desire to estimate the size of the limbs of the Mandelbrot set.

THEOREM D. (The PLY-inequality for a hyperbolic or parabolic domain.) *Let Λ be a simply connected n-periodic, hyperbolic or parabolic Fatou domain. Then there exists a constant M_Λ depending only on the restriction $R^n : \Lambda \to \Lambda$ such that for any periodically accessible boundary point $\alpha \in \partial\Lambda$, with period $k \in \mathbb{N}$, multiplier $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$ and rotation number p/q, there exists a logarithm $L \in \mathbb{H}_+$ of λ, with*

$$|L - (p/q)2\pi i| \leq \frac{2k\sin\theta}{qnN} M_\Lambda,$$

where θ is the angle between $2\pi i$ and $L - (p/q)2\pi i$, and where N is the number of cycles of periodic accesses to α represented in Λ. Moreover if Λ is hyperbolic, then $M_\Lambda \leq (\log d + D)$, where d is the degree of the restriction $R^n : \Lambda \to \Lambda$ and D is the maximal hyperbolic distance in Λ between the attracting fixed point for R^n in Λ and its pre-images in Λ.

COROLLARY D.1. (Yoccoz, 1987.) *Let P be a polynomial of degree d, with connected Julia set. Let α be a repelling periodic point for P, with period $k \in \mathbb{N}$, multiplier $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$ and rotation number p/q. Then there exists a logarithm $L \in \mathbb{H}_+$ of λ, with*

$$|L - (p/q)2\pi i| \leq \frac{2k\sin\theta}{qN} \log d,$$

where θ is the angle between $L - (p/q)2\pi i$ and $2\pi i$, and N is the number of cycles of periodic rays landing at α.

COROLLARY D.2. *Let (U, U', f) be a polynomial-like mapping of degree d, with connected filled Julia set. Denote by m the modulus of the annulus $U\backslash K_f$. Let α be a repelling periodic point for f, with period $k \in \mathbb{N}$, multiplier $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$ and rotation number p/q, then there exists a logarithm $L \in \mathbb{H}_+$ of λ, with*

$$|L - (p/q)2\pi i| \leq \frac{2k\sin\theta}{qN}\left(\log d + \frac{d\pi}{2m}\right),$$

where θ is the angle between $L - (p/q)2\pi i$ and $2\pi i$, and N is the number of cycles of periodic accesses $(\alpha, [\gamma])$ in $U\backslash K_f$.

Levin has proved a version of Corollary D.2 in his paper. His version being without rotation number and with the factor $K \log d$ instead of $(\log d + d\pi/2m)$, where K is the smallest possible constant of quasiconformality for a hybrid equivalence between (U, U', f) and a polynomial. Douady has encouraged us to give an inequality in terms of the modulus of the annulus $U \setminus K_f$, as the number K is not well controlled, at the present stage of the theory.

Remark. Geometrically the inequalities indicate that the appropriate logarithm L of λ is contained in the disc with center $r + i2\pi p/q$ and radius r, where r is the right-hand side of the inequalities divided by $2\sin\theta$. Hence $\lambda \in \exp(\mathbb{D}(r + i2\pi p/q, r))$ (see Figure 1).

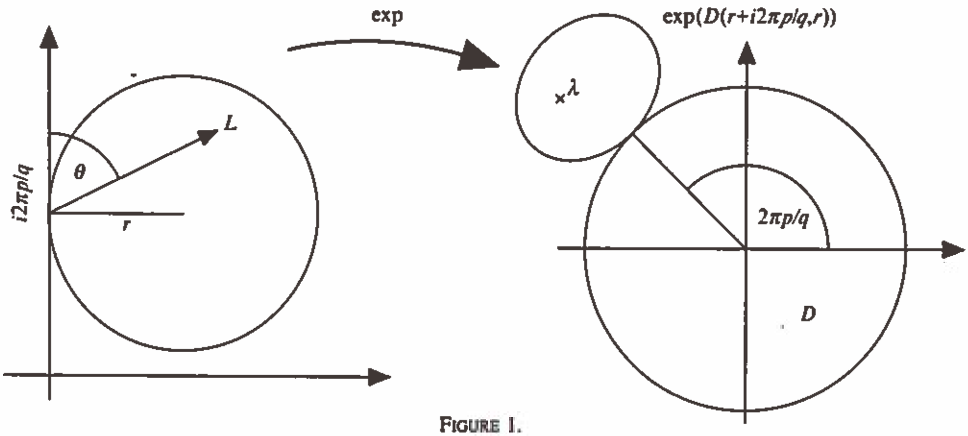

FIGURE 1.

2. *Proof of Theorem A*

Accesses. For completeness we shall briefly summarize the main definitions and results concerning accesses. An introduction to accesses can be found in the monograph by Goluzin [G].

Let $U \subseteq \bar{\mathbb{C}}$ be an open connected subset. A boundary point $\alpha \in \partial U$ is said to be an *accessible boundary point* of U, if there exists a curve $\gamma : [0, 1] \to \bar{U}$ with $\gamma(0) = \alpha$ and $\gamma(]0, 1]) \subseteq U$. Such a curve γ is said *to access* α. Suppose further that U is finitely connected, i.e. the complement of U has a finite number of connected components. Two curves γ_0 and γ_1 accessing α are termed *equivalent*, iff for any neighbourhood Δ of α, there exists a curve $\kappa : [0, 1] \to \Delta \cap U$ with $\kappa(0) \in \gamma_0$ and $\kappa(1) \in \gamma_1$. An *access* to α in U is a pair $(\alpha, [\gamma])$, where α is an accessible boundary point of U and $[\gamma]$ is an equivalence class of curves in U accessing α. A *crosscut* in U is an arc $c : [0, 1] \to \bar{U}$ such that $c(0)$ and $c(1)$ are distinct boundary points of U and $c(]0, 1[) \subseteq U$.

THEOREM. *Let $U \subset \bar{\mathbb{C}}$ be an open set isomorphic to $\mathbb{D}$ and let $\phi : U \to \mathbb{D}$ be a conformal equivalence, then:*

(a) *for each access $(\alpha, [\gamma])$ there exists a unique point $\alpha' \in \mathbb{S}^1$ such that for each $\gamma' \in [\gamma]$ $\phi \circ \gamma'(s) \to \alpha'$ as $s \to 0$.*

(b) *Two distinct accesses in U correspond to two distinct points in $\mathbb{S}^1$.*

(c) *The set of points in $\mathbb{S}^1$ corresponding to accesses in U is dense in $\mathbb{S}^1$.*

Proof. [G, p. 37; Theorem 1]. This proof works if U is bounded in $\mathbb{C}$. In the general case one can apply Möbius transformations and a square root on U to obtain a bounded subset of $\mathbb{C}$. □

If U is finitely connected, but not simply connected, the conclusions of Theorem 1 still hold, provided we substitute for $\mathbb{D}$ an open bounded subset of $\mathbb{C}$, conformally equivalent to U, with analytic Jordan curves as boundary components. For a discussion of this case see [G, pp 205–208], which treats the doubly connected case and shows the idea for the general case.

Suppose U is an open connected subset of $\bar{\mathbb{C}}$. Define two curves γ_0, γ_1 accessing $\alpha \in \partial U$ from U, to be homotopy equivalent, if there exists a homotopy of γ_0 to γ_1 through curves accessing α from U. It is an easy exercise to show that if U is also finitely connected, then the two notions of equivalence of accessing curves are equivalent.

Linearizing maps. Let f be a holomorphic map defined in a neighbourhood of a point α. Suppose α is a fixed point for f with multiplier λ, i.e. $f(\alpha) = \alpha$, and $f'(\alpha) = \lambda$. If α is repelling, i.e. $|\lambda| > 1$, then there exists a local linearizing parameter $\psi : \mathbb{D}(r) \to \Delta(r)$, for f at α, i.e. ψ is univalent with $\psi(0) = \alpha$ and

$$f \circ \psi(z/\lambda) = \psi(z), \quad \forall z \in \mathbb{D}(r). \tag{2.1}$$

Imposing the extra condition $\psi'(0) = 1$, determines the germ of ψ at 0 uniquely and we denote by $\Delta(t)$ the sets $\psi(\mathbb{D}(t))$, for $0 < t \leq r$. If α is parabolic, i.e. $\lambda = \exp(i2\pi p/q)$, $(p, q) = 1$ and if f is not of order q, i.e. $f^q \not\equiv \mathrm{Id}$, then there exists an integer $\nu \geq 1$ such that the multiplicity of α as a fixed point for f^q is $\nu q + 1$. We can suppose wlog that $\alpha = 0$ and thus $f^q(z) = z(1 + z^{\nu q}) + O(z^{\nu q+1})$ conjugating by a linear map, if necessary. There exists a system of $2\nu q$ Fatou coordinates for f around 0: for r sufficiently small and $0 \leq j < \nu q$ there exist neighbourhoods Δ_j, Ω_j of the segments $]0, re^{i\pi 2j/\nu q}]$ and $]0, re^{i\pi(2j+1)/\nu q}]$ respectively, and biholomorphic maps, local Fatou coordinates, $\psi_j : \Delta_j \to \mathbb{H}_-$, $\phi_j : \Omega_j \to \mathbb{H}_+$ with:

$$f^q \circ \psi_j^{-1}(z - 1) = \psi_j^{-1}(z), \quad \forall z \in \mathbb{H}_- \tag{2.2}$$

$$\phi_j \circ f^q(z) = \phi_j(z) + 1, \quad \forall z \in \Omega_j, \tag{2.3}$$

where $\mathbb{H}_\pm = \{z | \Re(z) \gtrless 0\}$. The germs of the local linearizing maps ψ_j and ϕ_j are unique up to post-composition with translations, here we take germ of say ϕ_j to mean the restriction to any sectorial neighbourhood with positive opening angle to both sides of a semi-open segment $]0, te^{i\pi(2j+1)/\nu q}]$, $0 < t \leq r$. The domains of ψ_j and ϕ_j are, on the contrary, not unique. Each Δ_j and Ω_j has opening angle $\pi/\nu q$ towards 0 centered at $\exp(i\pi 2j/\nu q)$, resp. $\exp(i\pi(2j+1)/\nu q)$ i.e. the boundary of say Δ_0 is a Jordan curve $\kappa : [0, 1] \to \mathbb{C}$, $\kappa(0) = \kappa(1) = 0$, which emanates from 0 in the direction tangentially to $\exp(-i\pi/2\nu q)$ and returns to 0 tangentially to the direction $\exp(i\pi/2\nu q)$. The local Fatou coordinates have univalent extensions to domains $\Delta_j' \supset \Delta_j$ resp. $\Omega_j' \subset \Omega_j$ having twice the opening angles and being centered on the same axes. Furthermore the Δ_j' and Ω_j' can be chosen so that $\overline{\Delta_j'} \subset f^q(\Delta_j') \cup \{0\}$ and $\overline{f^q(\Omega_j')} \subset \Omega_j' \cup \{0\}$ (see **[DH-2**, expose IX]).

Periodic accesses. We shall show some fundamental properties of periodic accesses. We remind the reader that if Λ is a simply connected n-periodic Fatou domain for R and $\phi : \Lambda \to \mathbb{D}$ is a Riemann map, then the Blaschke product R_Λ, whose restriction to $\mathbb{D}$ is $R_\Lambda = \phi \circ R^n \circ \phi^{-1}$, is called a conjugate map for R^n on Λ. Note that trivially a periodic access corresponds under ϕ to a unique periodic point for R_Λ in $\mathbb{S}^1$. Among other things Proposition 2.1 shows that the injective map from the set of periodic accesses in Λ to the set of periodic points for R_Λ in $\mathbb{S}^1$ is also surjective. Also Proposition 2.1 can be viewed as a generalization to simply connected parabolic and hyperbolic domains of ([**D–H-1** Proposition 2, pp 70–72]). Parts of Proposition 2.1 have been proved by Pommerenke ([**Po**, Theorem 1] in a different terminology). One should note however that the last statement in Pommerenke's theorem is not true, as the example $P(z) = z^2 + 1/4$ shows. The proof we present here is due to Sullivan, Douady and Hubbard. An access $(\alpha, [\gamma])$ is called *preperiodic* if there exists a $l \geq 0$ such that the access $R^l(\alpha, [\gamma])$ is periodic. The point α is then preperiodic and is said to be *preperiodically accessible*. For brevity we shall use the notation $I_r =]0, r]$, $r > 0$.

PROPOSITION 2.1. *Let Λ be a simply connected n-periodic parabolic or hyperbolic Fatou domain for R, with conjugate map R_Λ. Let $\alpha' \in \mathbb{S}^1$ be a (pre)periodic point for R_Λ. Then α' corresponds to a (pre)periodic access $(\alpha, [\gamma])$ of Λ. The periodic points in the orbit of α are either repelling or parabolic. If parabolic and of period k, their common multiplier is an (nk'/k)th root of unity, where k' is the eventual period of α'.*

Proof.
Case 1. *α' is periodic under R_Λ with period $k' \in \mathbb{N}$.* The multiplier λ' of α' is real and satisfies $\lambda' \geq 1$, because $R_\Lambda(\mathbb{D}) = \mathbb{D}$, $R_\Lambda(\mathbb{S}^1) = \mathbb{S}^1$ and $\mathbb{D}$ is either a parabolic or hyperbolic domain for R_Λ. We distinguish two subcases, first if $\lambda' = 1$, then $\mathbb{D}$ and Λ are connected components of the immediate attracted basins for parabolic points α' and α for R_Λ and R respectively. Let $\phi_0 : \Omega_0 \subset \mathbb{D} \to \mathbb{H}_+$ be a local Fatou coordinate for R_Λ. Define $\hat{\gamma}_{\alpha'} : I_1 \to \mathbb{D}$ by $\hat{\gamma}_{\alpha'}(t) = \phi_0^{-1}(-\log t)$ and $\gamma_{\alpha'} = \phi^{-1} \circ \hat{\gamma}_{\alpha'}$. Then $\hat{\gamma}_{\alpha'}(t) \to \alpha'$ and $\gamma_{\alpha'}(t) \to \alpha$ as $t \to 0$. We define $\gamma_{\alpha'}(0) = \alpha$. Then $\gamma_{\alpha'}$ defines an n periodic access to α, corresponding to α'. To conclude the subcase we note that the snail Lemma [**M**, 13.2] implies that $(R^n)'(\alpha) = 1$. Thus the multiplier of α is an nk'/kth root of unity, since the period k' of α' equals 1.

Next we suppose $\lambda' > 1$ and thus α' is repelling. Let $\psi : \mathbb{D}(r) \to \Delta(r)$ be a local linearizing parameter for $R_\Lambda^{k'}$ at α' and let $\Sigma = \mathbb{D}(r) \cap \mathbb{H}_+$. Composing with multiplication by some number in $\mathbb{S}^1$, if necessary, we can suppose that $\psi(\Sigma) \subset \mathbb{D}$. Let $\kappa : I_r \to \Sigma$ be given by $\kappa(t) := t$. Then $\forall t \in I_r : \lambda' \cdot \kappa(t/\lambda') = \kappa(t)$. Let $\gamma_{\alpha'} : I_r \to \Lambda$ denote the arc $\phi^{-1} \circ \psi \circ \kappa$. Then $R^{k'n}(\gamma_{\alpha'}(t/\lambda')) = \gamma_{\alpha'}(t)$, $\forall t \in I_r$. For use in the non-periodic case, Case 2, we also define the arc $\hat{\gamma}_{\alpha'} = \phi \circ \gamma_{\alpha'} : I_r \to \mathbb{D}$. Note that trivially $\hat{\gamma}_{\alpha'}(t) \to \alpha'$ as $t \to 0$. (See Figure 2.)

Let $\omega = \phi^{-1} \circ \psi(\Sigma)$ and $\omega' = \phi^{-1} \circ \psi(\Sigma/\lambda') \subsetneq \omega$. Then the restriction $R^{nk'} : \omega' \to \omega$ is biholomorphic and thus expanding with respect to the hyperbolic metric on ω. There exists $L > 0$ such that $d(z, R^{nk'}(z)) \leq L$ for all z in say $\gamma_{\alpha'}(]0, r/2\lambda'])$, where $d(\cdot, \cdot)$ denotes distance with respect to the hyperbolic metric on ω. This is because of the expansiveness of $R^{nk'}$ and the invariance of $\gamma_{\alpha'}$ under $R^{nk'}$. Any limit point of $\gamma_{\alpha'}(t)$

for $t \to 0$ is a fixed point of $R^{nk'}$, because $\gamma_{\alpha'}(t) \to \partial\omega$ as $t \to 0$ and the Euclidean diameter of the hyperbolic ball of fixed radius L around z tends to 0 as z approaches the boundary. Thus $\gamma_{\alpha'}$ converges to a fixed point α for $R^{nk'}$, since the fixed points of $R^{nk'}$ are isolated and the limit points of $\gamma_{\alpha'}(t)$ for $t \to 0$ form a connected set. Define $\gamma_{\alpha'}(0) = \alpha$, then clearly $\gamma_{\alpha'}$ defines a periodic access to α corresponding to α' and α is either repelling or indifferent. If α is indifferent, then the snail Lemma [M, 13.2] implies that $(R^{nk'})'(\alpha) = 1$. Thus the multiplier of α is an nk'/k-root of unity, where k is the period of α. This completes the periodic case.

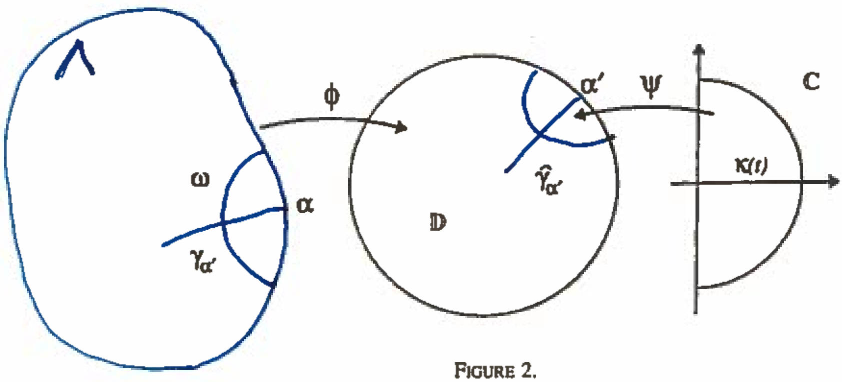

FIGURE 2.

Case 2. *The point α' is strictly preperiodic.* Choose $m \in \mathbb{N}$ such that $\beta' := R^m_\Lambda(\alpha')$ is periodic for R_Λ. Consider the periodic arcs $\hat{\gamma}_{\beta'} : I_r \to \mathbb{D}$ and $\gamma_{\beta'} : I_r \to \Lambda$ from the periodic case above. Taking r smaller if necessary, there exists a unique lift $\hat{\gamma}_{\alpha'} : I_r \to \mathbb{D}$ of $\hat{\gamma}_\beta : I_r \to \mathbb{D}$ to R^m_Λ with $\hat{\gamma}_{\alpha'}(s)$ tending to α' as $s \to 0$.

Let $\gamma_{\alpha'} := \phi^{-1} \circ \hat{\gamma}_{\alpha'} : I_r \to \Lambda$, then $\forall s \in I_r : R^{nm} \circ \gamma_{\alpha'}(s) = \gamma_{\beta'}(s)$. From the periodic case we know that, $\gamma_{\beta'}(s)$ converges to a periodic point β as $s \to 0$. Hence it follows from the theorem of analytic continuation that $\gamma_{\alpha'}(s)$ tends to a single point α, a pre-image of β under R^{nm} as $s \to 0$. If we define $\gamma_{\alpha'}(0) = \alpha$, then clearly $\gamma_{\alpha'}$ defines a preperiodic access to α corresponding to α'. □

Complement of Proposition 2.1. The Riemann map ϕ defines a natural bijective map between the set of (pre)periodic accesses in Λ and the set of (pre)periodic points for R_Λ in $\mathbb{S}^1$. Furthermore let $(\alpha, [\gamma])$ be an m-periodic access of Λ. Then $[\gamma]$ contains an arc, the center curve, $\gamma_{\alpha'} : [0, r] \to \Lambda \cup \{\alpha\}$ with $R^m(\gamma_{\alpha'}) \subset \gamma_{\alpha'}$ or $\gamma_{\alpha'} \subset R^m(\gamma_{\alpha'})$.

The periodic arcs of the above complement and their preperiodic pre-images in Λ can be viewed as generalizations of rational external rays of polynomials: suppose the periodic point α' for R_Λ, corresponding to the curve $\gamma_{\alpha'}$ in the above complement, is repelling. Then $\gamma_{\alpha'}$ extends to a curve defined on all of $\mathbb{R}_+ \cup \{0\}$ by use of the periodicity $R^{k'n}(\gamma_{\alpha'}(t/\lambda')) = \gamma_{\alpha'}(t)$, $\forall t \in I_r$. The extended curve converges, as the parameter tends to ∞, to an attracting periodic point z_0 in Λ, if Λ is hyperbolic, or a parabolic periodic point $z_0 \in \partial\Lambda$, if Λ is parabolic. If $R_\Lambda(z) = z^d$, then γ is a ray, i.e. $\gamma = \phi^{-1}(]0, \alpha'[)$.

A continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ will be called vaguely expanding if, for any periodic point x_0 for f, say of period $m \in \mathbb{N}$, there exists a neighbourhood $\omega \subseteq \mathbb{S}^1$ of x_0 such that

$$\forall x \in \omega : \quad d(x_0, x) < d(x_0, f^m(x)),$$

where $d(\cdot,\cdot)$ denotes distance with respect to the standard metric on $\mathbf{S}^1$. For a parabolic or hyperbolic domain Λ the restriction to $\mathbf{S}^1$ of the conjugate map is vaguely expanding.

LEMMA 2.2. *Let $f : \mathbf{S}^1 \to \mathbf{S}^1$ be vaguely expanding and let $x_0 \in \mathbf{S}^1$. Let $\{x_m\}_{m=0}^{\infty}$ be the sequence of iterates of x_0, i.e. $x_m = f^m(x_0)$ for all $m \geq 0$. Suppose the sequence is a monotonous sequence in some sub-interval $I \subset \mathbf{S}^1$. Then x_0 is prefixed under f.*

Proof. Since the sequence $\{x_m\}_{m=0}^{\infty}$ is a monotonous sequence in a subinterval I of $\mathbf{S}^1$, there exists a point $x_\infty \in \bar{I}$ such that $x_m \to x_\infty$ as $m \to \infty$. The point x_∞ is a fixed point for f, since f is continuous. If there exists an $l \in \mathbb{N}$ with $x_l = x_\infty$, then x_0 is prefixed by f. If not, then the x_m come arbitrarily close to x_∞ and for each $l \in \mathbb{N} : x_{l+1} \in [x_l, x_\infty] \subseteq I$. This contradicts that f is vaguely expanding. Hence x_0 is pre-fixed. □

Proof of Theorem A. First we show that any interior access of Λ to α is periodic. To this end we shall assume wlog that α, Λ and $(\alpha, [\kappa])$ are fixed by R. Let $(\alpha, [\gamma])$ be an access in Λ. Choose a representative γ_0 of $[\gamma]$ and for each $l \in \mathbb{N}$ let the curve γ_l be given by $\gamma_l := R^l \circ \gamma_0$. Further for each $l \in \mathbb{N}_0$ let x_l be the point in $\mathbf{S}^1$ corresponding to $(\alpha, [\gamma_l])$. Then $R_\Lambda(x_{l-1}) = x_l$ for all $l \in \mathbb{N}$.

Suppose $(\alpha, [\gamma_0])$ is not preperiodic, then for $l \in \mathbb{N}_0$ the curve γ_l neither belongs to $[\kappa]$ nor to $[\gamma_{l'}]$ for any $l' \in \mathbb{N}_0$ with $l \neq l'$. Furthermore R fixes the (exterior or interior) access $(\alpha, [\kappa])$ of Λ and R is injective on a small neighbourhood of α. Hence the curves $\{\gamma_l\}_{l=0}^{\infty}$ are distributed monotonically around α. But then the sequence $\{x_l\}_{l=2}^{\infty}$ is a monotonous non-preperiodic sequence in one of the two open sub intervals of $\mathbf{S}^1$ bounded by x_0 and x_1. This contradicts Lemma 2.2, since R_Λ is vaguely expanding. Hence the access $(\alpha, [\gamma_0])$ is preperiodic. Finally any preperiodic access $(\alpha, [\gamma_0])$ is periodic, because R is locally a homeomorphism at α.

Next we easily deduce statements (a), (b) and (c). The map R^{kq} fixes the (exterior or interior) access $(\alpha, [\kappa])$. Let $(\alpha, [\gamma])$ be any (interior) access of Λ to α. If $(\alpha, [\gamma])$ is not fixed by R^{kq}, then it can never be a periodic access of Λ, since at α the map R^{kq} is locally injective and fixes $(\alpha, [\kappa])$. Thus the period of $(\alpha, [\gamma])$ divides the period of $(\alpha, [\kappa])$. Exchanging the roles of γ and κ we also find that the period of $(\alpha, [\kappa])$ divides the period of $(\alpha, [\gamma])$, thus the two periods are equal. This proves (a) from which (b) immediately follows. According to (a) every access to α from Λ is qk-periodic. In addition every qk-periodic access to α in Λ corresponds to a unique qk/n-periodic point for R_Λ in $\mathbf{S}^1$. There is a finite number of such points. This proves (c). □

3. *Proof of Theorem B*

Moduli and Grötzsch inequalities. We shall need the notion of moduli of quadrilaterals and annuli together with two Grötzsch inequalities. We shall only give the facts needed here. The interested reader can find more details in [L–V].

A quadrilateral is a triple (Q, c_1, c_2) consisting of a simply connected open subset Q of $\bar{\mathbb{C}}$ and two disjoint boundary arcs $c_i : [0, 1] \to \partial Q$, $i = 1, 2$, which have the following properties: (a) The sets $\partial Q \backslash c_i(]0,1[)$ are connected. (b) No point of $c_i(]0,1[)$ is an accumulation point of $\partial Q \backslash c_i(]0,1[)$. Whenever it is clear what the boundary arcs are they will be omitted. We let Γ_Q denote the set of rectifiable curves in Q, connecting the two boundary arcs (i.e. curves $\gamma :]a,b[\to Q$ admitting a continuous extension with $\gamma(a) \in c_1$ and $\gamma(b) \in c_2$). For an annulus $A \subset \bar{\mathbb{C}}$ we let Γ_A denote the set of rectifiable curves in A connecting the two boundary components of A. A quadrilateral (Q', c_1', c_2') is a *subquadrilateral* of the quadrilateral (Q, c_1, c_2), if $Q' \subseteq Q$ and the boundary arcs c_1' and c_2' are subarcs of c_1 and c_2 one in each. Subquadrilaterals of an annulus A are defined likewise.

Definition 3.1. A quadrilateral (Q', c_1', c_2') is said to be *contained* in the quadrilateral (Q, c_1, c_2), if there exists a connected component T of $Q \cap Q'$ such that any curve $\gamma \in \Gamma_{Q'}$ has a restriction $\gamma_{|]a,b[} \in \Gamma_Q$ with $\gamma(]a,b[) \subset T$. The quadrilateral (Q', c_1', c_2'), is said to be contained in the annulus A, if the above properties are satisfied with (Q, c_1, c_2) replaced by A. The set T is called a *torso* of Q' with respect to Q (resp. A). (Torsos are not in general unique, as Figure 3 shows.)

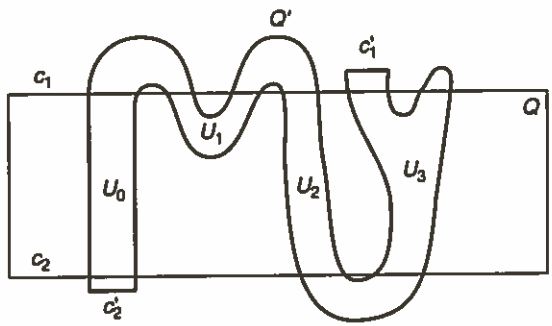

FIGURE 3. The quadrilateral (Q', c_1', c_2') is contained in the quadrilateral (Q, c_1, c_2). The sets U_0, U_2 and U_3 are torsos of Q' with respect to Q, U_1 is not.

Let Q be a quadrilateral. For a conformal Borel(-measurable) metric ρ on Q let $l_\rho(\cdot)$ denote arclength with respect to ρ and $\mathrm{Area}(Q, \rho)$ denote the area of Q with respect to ρ. A metric ρ is called admissible for Γ_Q, if ρ is a conformal Borel metric on Q and $l_\rho(\gamma) \geq 1$ for all $\gamma \in \Gamma_Q$. The modulus of Q is a conformal invariant of Q defined by $\mathrm{mod}(Q) := \inf\{\mathrm{Area}(Q, \rho) | \rho \text{ admissible}\}$. Suppose Q has modulus m, then there exists a univalent map ϕ_Q from Q onto the rectangle $]0, m[\times]0, i[$, such that ϕ_Q extends to a homeomorphism between Q union its boundary arcs and the rectangle union its horizontal boundaries. The map ϕ_Q is called a canonical map on Q and it is extremal in the following sense: $\mathrm{mod}(Q) = \mathrm{Area}(Q, |\phi_Q'|) < \mathrm{Area}(Q, \rho)$, where ρ is any other admissible metric on Q. The admissible metrics on an annulus $A \subseteq \bar{\mathbb{C}}$ is defined analogously, while the modulus is defined by $(\mathrm{mod}(A))^{-1} := \inf\{\mathrm{Area}(Q, \rho) | \rho \text{ admissible}\}$.

A canonical map on an annulus A is a conformal equivalence $\phi_A : A \to A(r_1, r_2)$, where $A(r_1, r_2) := \{z \in \mathbb{C} | r_1 < |z| < r_2\}$.

LEMMA 3.2. *(Grötzsch.) Let Q be a quadrilateral containing the family of quadrilaterals*

$\{Q_j\}_{j\in J}$. *Suppose there exists a family of corresponding disjoint torsos* $\{T_j\}_{j\in J}$, *then:*

$$\sum_{j\in J} \mathrm{mod}(Q_j) \le \mathrm{mod}(Q).$$

Equality is attained if and only if each Q_j *is a subquadrilateral of* Q, *whose image, under the canonical map* ϕ_Q *on* Q, *is a rectangle and* $\bigcup_{j\in J} \overline{Q_j} = \bar{Q}$.

Proof. Let ϕ_Q be a canonical map on Q and let $\rho := |\phi_Q'(z)|$ be the extremal admissible metric on Q. For each $j \in J$ let ρ_j be the conformal Borel metric on Q_j, which equals ρ on T_j and is zero elsewhere. Then ρ_j is an admissible metric on Q_j and

$$\sum_{j\in J} \mathrm{mod}(Q_j) \le \sum_{j\in J} \mathrm{Area}(Q_j, \rho_j) = \sum_{j\in J} \mathrm{Area}(T_j, \rho) \le \mathrm{Area}(Q, \rho) = \mathrm{mod}(Q).$$

Equality is attained in the first inequality if and only if each ρ_j is equal to $|\phi_j'|$, where ϕ_j is a canonical map on Q_j, this yields the first condition. Finally equality is obtained in the last inequality if and only if the last condition of the Lemma is fulfilled. □

LEMMA 3.3. *(Grötzsch.) Let* A *be an annulus containing the family of quadrilaterals* $\{Q_j\}_{j\in J}$. *Suppose there exists a family of corresponding disjoint torsos* $\{T_j\}_{j\in J}$, *then:*

$$\sum_{j\in J} \mathrm{mod}(Q_j) \le \frac{1}{\mathrm{mod}(A)}.$$

Equality is attained if and only if each Q_j *is a subquadrilateral of* A, *whose image, under the canonical map* ϕ_A *on* A, *is a sector bounded by two radial segments and* $\bigcup_{j\in J} \overline{Q_j} = \bar{A}$.

Proof. The proof is a simple copy of the proof of Lemma 3.2. □

LEMMA 3.4. *Let* $r \in]0, 1[$ *and* $\sigma \in \mathbb{S}^1$. *Let* Q *be any quadrilateral with* $Q \subseteq \mathbb{D}^*\setminus\overline{\mathbb{D}(\sigma, r)}$, *separating* 0 *from* $\mathbb{D}(\sigma, r) \cap \mathbb{D}$ *in* $\mathbb{D}$ *and whose boundary arcs are subarcs of* $\mathbb{S}^1$. *Then*

$$\mathrm{mod}\,(Q) \le \frac{1}{\pi} \cdot \log(4/r).$$

Proof. We can assume $\sigma = 1$ wlog. The pre-images of Q under $z \mapsto z^2$ are two disjoint quadrilaterals conformally equivalent to Q. They are sub-quadrilaterals of the quadrilateral, whose set is $\mathbb{D}\setminus(\overline{\mathbb{D}(-1, r/2)} \cup \overline{\mathbb{D}(1, r/2)})$, since the norm of the derivative of $z \mapsto z^2$ is bounded by 2 on $\bar{\mathbb{D}}$. The latter quadrilateral is conformally equivalent under $z \mapsto (1+z)/(1-z)$ to a subquadrilateral of the quadrilateral, $Q' := \mathbb{H}_+ \cap (\mathbb{D}(4/r)\setminus\overline{\mathbb{D}(r/4)})$ whose boundary arcs are the segments of $i \cdot \mathbb{R}$ on its boundary. Hence we obtain:

$$2\mathrm{mod}(Q) \le \tfrac{1}{\pi} \log(4/r)^2.$$ □

Let U and W be open simply connected subsets of $\bar{\mathbb{C}}$, such that U isomorphic to $\mathbb{D}$ and the boundary of W is a Jordan curve $\gamma : \mathbb{S}^1 \to \bar{\mathbb{C}}$. Suppose W intersects U, but does not contain U and let $b \in U\setminus\bar{W}$. Let X be a connected component of $W \cap U$. Then there exists a unique subarc c of γ with $c \subset \partial X$, such that c is a crosscut in U separating X from b (look at $\phi(X)$, where $\phi : U \to \mathbb{D}$ is a Riemann map). Let ω denote the unique connected component of $U\setminus c$ containing X. The triple (X, c, ω) will be called a W *piece* of U with respect to b. Usually the point b will be omitted, when it is clear from the context which point is meant.

Proof of Theorem B. Let Λ be a periodic Fatou domain and let $\alpha \in \partial\Lambda$ be a repelling or parabolic periodic point for R. Assume that α does not belong to the closure of a non-periodic connected component of $R^{-m}(\Lambda)$ for any $m > 0$. Taking a sufficiently high iterate of R, we can suppose that both Λ and α are fixed by R. Then by assumption α belongs to the closure of no other connected component of $R^{-1}(\Lambda)$.

We shall suppose Λ to be simply connected. The other case of Λ being doubly connected and thus an Arnold–Herman ring can be treated analogously. Let $\phi : \Lambda \to \mathbb{D}$ be a Riemann map and let R_Λ be the conjugate map on Λ.

Let Λ' be the subset of Λ given by the following. If R_Λ is hyperbolic we can assume wlog, that 0 is a fixed point for R_Λ. Choose a disc $\mathbb{D}(r_1)$, $0 < r_1 < 1$, which contains all the critical values for R_Λ in $\mathbb{D}$. Define $\Lambda' := \phi^{-1}(\mathbb{D}(r_1))$. If R_Λ is parabolic we can assume wlog, that 1 is a parabolic fixed point for R_Λ. Choose a horor disc with root at 1, $\mathbb{D}(1 - r_1, r_1)$, $1/2 < r < 1$, containing all the critical values of R_Λ in $\mathbb{D}$ and define $\Lambda' := \phi^{-1}(\mathbb{D}(1 - r_1, r_1))$. Finally if Λ is a Siegel disc let Λ' be the center of the Siegel disc. Let

$$\Lambda'' := \Lambda \cap R^{-1}(\Lambda \backslash \bar{\Lambda}'),$$

then $\Lambda'' \subset \Lambda \backslash \bar{\Lambda}'$ and the restriction $R_| : \Lambda'' \to \Lambda \backslash \bar{\Lambda}'$ is a covering map.

Consider first the case where α is repelling with multiplier λ. Let $\psi : \mathbb{D}(r) \to \Delta(r)$ be a local linearizing parameter for R at α. Choose $\tau \in]0, r[$ so small that $\Delta(\tau) \cap \bar{\Lambda}' = \emptyset$ and no connected component of $R^{-1}(\Lambda)$ different from Λ intersects $\Delta(\tau)$. This is possible, because $\bar{\Lambda}'$ is a compact set not containing α and there is only a finite number of other (non-periodic) connected components of $R^{-1}(\Lambda)$ none of which has α in its closure.

Let $U_{0,0}$ be a connected component of $\Delta(\tau) \cap \Lambda$. The set $U_{0,0}$ has a unique pre-image $U_{m,m}$ under R^m in $\Delta(\tau/|\lambda|^m)$ for each $m \in \mathbb{N}$. Let $(U_{m,l}, c_{m,l}, \omega_{m,l})$ be the unique $\Delta(\tau/|\lambda|^l)$ piece of Λ containing $U_{m,m}$, for $l \in \{0, \ldots, m\}$. Then for each m we have nested sequences $U_{m,m} \subset \ldots \subset U_{m,0}$ and $\omega_{m,m} \subset \ldots \subset \omega_{m,0}$. We shall show the following:

Claim. $\exists m, m' \in \mathbb{N}$, $m \neq m'$ such that $U_{m,0} = U_{m',0}$.

Proof of Claim. It suffices to find $m, m' \in \mathbb{N}$ such that $U_{m,0} \cap U_{m',0} \neq \emptyset$, since both sets are connected components of the same set $\Delta(\tau) \cap \Lambda$. We remark that the claim then is trivial for a Siegel disc. The idea of the proof in the hyperbolic and parabolic cases is based on the following trivial observation. Given an $N \times N$ chessboard the modulus of a row is N^2 times that of a column.

For each $m \in \mathbb{N}$ and each $l \in \{1, \ldots, m\}$ let $Q_{m,l}$ be the quadrilateral, whose set is $\omega_{m,l-1} \backslash \bar{\omega}_{m,l}$ and whose boundary arcs are $c_{m,l-1}$ and $c_{m,l}$. Then for each $m \in \mathbb{N}$ the m quadrilaterals $\{Q_{m,l}\}_{l=1}^m$ are disjoint (look at the corresponding images under the Riemann map ϕ on Λ). Furthermore the quadrilateral $Q_{m,1}$ is contained in the annulus $\Delta(\tau) \backslash \overline{\Delta(\tau/|\lambda|)}$. Let T_m be the connected component of $U_{m,0} \backslash \overline{\Delta(\tau/|\lambda|)}$, with $c_{m,0}$ as a boundary arc. Then T_m is a torso of $Q_{m,1}$ with respect to $\Delta(\tau) \backslash \overline{\Delta(\tau/|\lambda|)}$. (see Figure 4.)

Suppose the claim is false, so that the sets $\{U_{m,0}\}_{m>0}$ are mutually disjoint. Then the torsos $\{T_m\}_{m>0}$ are mutually disjoint. The Grötszch's inequality for quadrilaterals in an

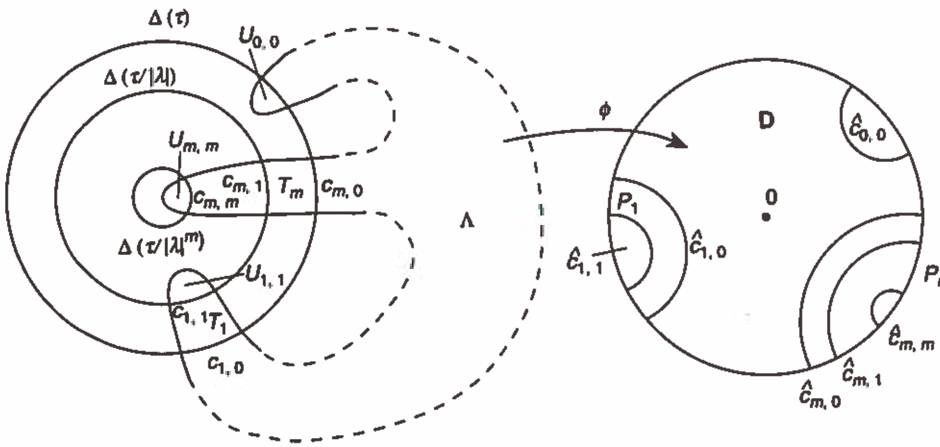

FIGURE 4.

annulus, Lemma 3.3, then yields

$$\sum_{m>0} \operatorname{mod}(Q_{m,1}) \leq \frac{1}{\operatorname{mod}(A)} = \frac{2\pi}{\log|\lambda|}. \tag{3.1}$$

On the other hand $\operatorname{mod}(Q_{m,l}) = \operatorname{mod}(Q_{m-l+1,1})$, because R^{l-1} maps $Q_{m,l}$ biholomorphically to $Q_{m-l+1,1}$, ($Q_{m-l+1,1} \subset \Lambda'$ is simply connected and $R : \Lambda'' \to \setminus\bar{\Lambda}'$ is a covering map).

Our strategy will be to estimate in two different ways the modulus of the 'big' quadrilaterals $(\omega_{m,0}\setminus\bar{\omega}_{m,m}, c_{m,0}, c_{m,m})$ and obtain a contradiction, when m is sufficiently big. We start by transferring the quadrilaterals to $\mathbb{D}$ using ϕ: $\Lambda \to \mathbb{D}$.

For each $m \in \mathbb{N}$ and for each $l \in \{1, \ldots, m\}$ let $\phi(Q_{m,l})$ be the quadrilateral in $\mathbb{D}$ conformally equivalent to $Q_{m,l}$ under ϕ. Then the boundary of the underlying set is a Jordan curve consisting of the two arcs $\phi(c_{m,l-1}), \phi(c_{m,l})$ and the two complementary arcs $a_{m,l}, b_{m,l}$, which are subarcs of $\mathbb{S}^1$. Let $P_{m,l}$ be the quadrilateral $(\phi(Q_{m,l}), a_{m,l}, b_{m,l})$. Further for each $m \in \mathbb{N}$ let P_m be the quadrilateral in $\mathbb{D}$, whose boundary arcs are on $\mathbb{S}^1$ and whose remaining boundary consists of the two curves $\phi(c_{m,0})$ and $\phi(c_{m,m})$. Then the m quadrilaterals, $P_{m,1}, \ldots, P_{m,m}$, are disjoint subquadrilaterals of the quadrilateral P_m, hence:

$$\operatorname{mod}(P_m) \geq \sum_{l=1}^{m} \operatorname{mod}(P_{m,l}) = \sum_{l=1}^{m} \frac{1}{\operatorname{mod}(Q_{m,l})} = \sum_{j=1}^{m} \frac{1}{\operatorname{mod}(Q_{j,1})}, \tag{3.2}$$

where we have used $\operatorname{mod}(Q_{m,l}) = \operatorname{mod}(Q_{m-l+1,1})$ and $\operatorname{mod}(P_{m,l}) \cdot \operatorname{mod}(\phi(Q_{m,l})) = 1$. We use (3.1), (3.2) and the Cauchy–Schwarz inequality, to obtain a lower bound for $\operatorname{mod}(P_m)$

$$\begin{aligned} \operatorname{mod}(P_m) \geq \sum_{j=1}^{m} \frac{1}{\operatorname{mod}(Q_{j,1})} &\geq \frac{\log|\lambda|}{2\pi} \left(\sum_{j=1}^{m} \operatorname{mod}(Q_{j,1}) \right) \left(\sum_{j=1}^{m} \frac{1}{\operatorname{mod}(Q_{j,1})} \right) \\ &\geq \frac{m^2 \log|\lambda|}{2\pi}. \end{aligned} \tag{3.3}$$

To obtain an upper bound for $\mathrm{mod}(P_m)$ we proceed as follows. For $m \in \mathbb{N}$ let Ω_m be the connected component of $\mathbb{D}\backslash\phi(c_{m,m})$, which does not contain 0. Then R_Λ maps Ω_{m+1} biholomorphically to Ω_m. Choose $z_1 \in \mathbb{S}^1$ and $r_0 > 0$ such that $(\mathbb{D}(z_1, r_0) \cap \mathbb{D}) \subseteq \Omega_1$ and let z_m be the unique point of $\bar{\Omega}_m$, which is mapped to z_1 by R_Λ^{m-1}. Then Ω_m contains the 'semi'-disc $\mathbb{D} \cap \mathbb{D}(z_m, r_0/M^{m-1})$ where $M = \max|R'_\Lambda(z)|$ for $|z| \leq 1$, and the quadrilateral P_m is contained in $\mathbb{D}^*\backslash\overline{\mathbb{D}(z_m, r_0/M^{m-1})}$. Thus P_m satisfies the requirements of Lemma 3.4 with $\sigma = z_m$ and $r = r_0/M^{m-1}$. Lemma 3.4 yields

$$\mathrm{mod}(P_m) \leq m\frac{\log(M)}{\pi} + \frac{1}{\pi}\log\left(\frac{4}{r_0 M}\right). \tag{3.4}$$

The two bounds (3.3) and (3.4) can both be true only up to a certain $m \in \mathbb{N}$, since one is linear and the other is quadratic in m. Thus if the claim is not true we obtain a contradiction.

We can assume $m > m'$. Let $l := m - m'$, then $U_{m,l} \subseteq \Delta(\tau/|\lambda|^l) \cap U_{m,0}$ and $U_{m,l}$ is mapped biholomorphically to $U_{m',0} = U_{m,0}$ by R^l. Let $f : U_{m,0} \to U_{m,l}$ denote the corresponding local inverse. Choose $z_0 \in U_{m,0}$ and let $z_1 := f(z_0) \in U_{m,l} \subset U_{m,0}$. Let $\gamma : [0, 1] \to U_{m,0}$ be a curve with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. We extend γ to a curve $\hat{\gamma} : \mathbb{R} \cup \{0\} \to U_{m,0}$ using f : for $t \geq 1$ let $j \in \mathbb{N}$ be such that $t - j \in [0, 1]$ and define $\hat{\gamma}(t) := f^j(\gamma(t - j))$. Then $\hat{\gamma}(t) \to \alpha$ as $t \to \infty$ and $\hat{\gamma}$ defines an l-periodic access to α.

This concludes the proof of the case α is repelling. In the other case where α is parabolic, we have to distinguish two subcases, namely the subcase when Λ is a connected component of the attracted basin of α and the subcase, where it is not. The first subcase is an immediate consequence of the existence of Fatou coordinates, even if we drop the second assumption in our theorem. The second subcase can be handled by essentially the same method as that used in the proof of the case α is repelling. We will therefore only point out the differences.

We can suppose that each attracted petal is fixed by R, and thus is mapped inside itself. Then also each repelled petal has a unique pre-image inside itself. Furthermore we can suppose, by assumption, taking smaller petals if necessary, that (a) Λ is the only connected component of $R^{-1}(\Lambda)$ which intersects the repelled petals (by assumption Λ is fixed by R), and (b) the repelled petals do not intersect Λ'.

Let Δ_0 be a repelled petal, which intersects Λ and let Δ_l, Δ_r be the two neighbouring attracted petals, then $\Lambda \cap \Delta_0$ is contained in $\Delta_0\backslash(\overline{\Delta_l} \cup \overline{\Delta_r})$. Let $f : \Delta_0 \to \Delta_0$ denote a local inverse of R and for each $m \in \mathbb{N}$ let $\Delta_m := f^m(\Delta_0)$. Then the boundary of each Δ_m is a Jordan curve. Let Q be the quadrilateral, whose set is $\Delta_0\backslash(\overline{\Delta_1} \cup \overline{\Delta_l} \cup \overline{\Delta_r})$ and whose boundary arcs are the segments of $\partial\Delta_0$ and $\partial\Delta_1$ on the boundary of Q.

Let $U_{0,0}$ be a connected component of $\Delta_0 \cap \Lambda$. Let $(U_{m,l}, c_{m,l}, \omega_{m,l})$ be the Δ_l piece of Λ, which contains $f^m(U_{0,0})$, for $m \in \mathbb{N}$ and $l \in \{0, \ldots, m\}$. From here and onwards the proof is identical to the proof in the repelling case, except that the right-hand side of equation (3.1) is substituted by $\mathrm{mod}(Q)$ in this case. □

Corollary B.1 is an immediate consequence of Theorem B, once we note that for a polynomial the attracted basin of infinity is connected and center curves of periodic accesses in $\Lambda(\infty)$ are external rays.

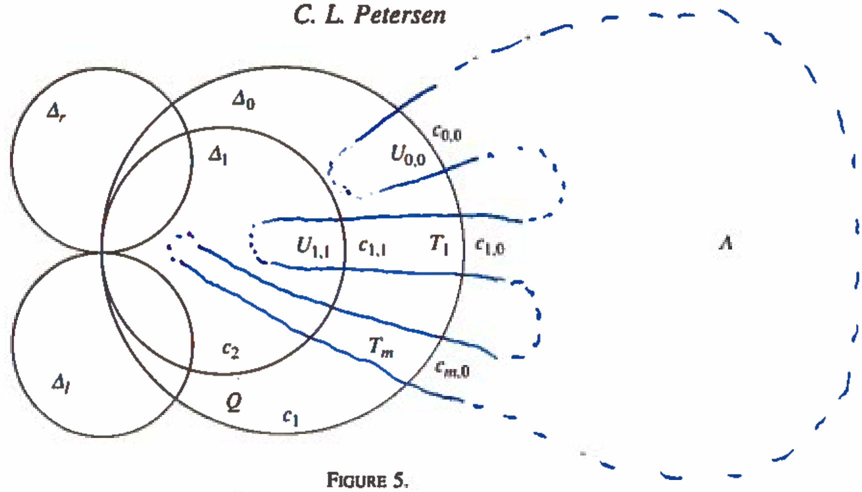

FIGURE 5.

4. *Proof of Theorems C and D*

Our proofs of Theorems C and D are based on a Grötszch inequality for non-trivial annuli in a torus. Let T be a torus isomorphic to $\mathbb{C}/\Gamma$, where $\Gamma := L \cdot \mathbb{Z} \oplus 2\pi i \cdot \mathbb{Z}$, $L \in \mathbb{H}_+$ and let $\Pi : \mathbb{C} \to T$ denote the corresponding universal covering. Note that also $T \simeq \mathbb{C}^*/(z \mapsto \lambda z)$, where $\lambda = \exp(L)$. To emphasize the dependence on L we often write T_λ for T. Also we let $\Pi_\lambda : \mathbb{C}^* \to T_\lambda$ denote the corresponding projection.

Suppose $\kappa : [0,1] \to T$, $\kappa(0) = \kappa(1)$ is a non-trivial (i.e. not homotopic to a constant) Jordan curve in the torus T and let $[\kappa]$ denote the homotopy class of κ in T. A metric ρ on T is called *admissible* for $[\kappa]$, if it is a conformal Borel metric and $l_\rho(\kappa') \geq 1$ for all $\kappa' \in [\kappa]$. The modulus of (T, κ) is a conformal invariant, defined by $\mathrm{mod}(T,\kappa) = \inf\{\mathrm{mes}(T,\rho) | \rho \text{ admissible}\}$. The modulus can be explicitly calculated, and it satisfies a Grötszch inequality (Lemma 4.1).

Given L, there exist $p, q \in \mathbb{Z}$, $(p,q) = 1$ such that any lift $\tilde{\kappa}$ of κ to Π satisfies

$$\tilde{\kappa}(1) = \tilde{\kappa}(0) + qL - p2\pi i.$$

We shall suppose that either $q > 0$ or $q = 0$ and $p = 1$, changing the orientation of κ if necessary. Further if $q > 0$ then adding $2\pi i$ to L, changes p to $p + q$. Thus for a suitable choice of L we have $p \in \{0, \ldots, q-1\}$. We call $\sigma = qL - p2\pi i$ the associated segment and the number p/q normalized as above the *(combinatorial) rotation number* of κ. The associated segment and the rotation number are homotopy invariants of κ. Furthermore if two non-trivial Jordan curves in T are disjoint, then after a change of orientation of one of them if necessary, the two curves are homotopic. Note that if $q > 0$, then the preimages of κ under $\Pi_\lambda : \mathbb{C}^* \to T = T_\lambda$ are q arcs from 0 to ∞ performing a p/q-rotation around 0 under multiplication by λ.

As above let $\Pi : \mathbb{C} \to T$ be a universal covering of the torus T. Let ρ be a flat metric on T, i.e. $\rho = \Pi_*(\hat{\rho})$, where $\hat{\rho}$ is a flat metric (proportional to the Euclidean) on $\mathbb{C}$. For U a Borel subset of T the quotient, $\mathrm{Area}(U,\rho)/\mathrm{Area}(T,\rho)$, does not depend on the choice of flat metric ρ on T, we call it the *relative flat area* and denote it by $\mathrm{rfa}(U)$.

LEMMA 4.1. *Let T be a torus and let κ be a Jordan curve in T with rotation number p/q, $q > 0$ and associated segment σ. Let $\{A_j\}_{j \in J}$ be any family of disjoint annuli homotopic* *to κ in T. Denote by $B := \sum_{j\in J} \mathrm{rfa}(A_j) \le 1$ the relative flat area of the family, then*

$$\sum_{j\in J} \mathrm{mod}(A_j) \le B\,\mathrm{mod}(T,\kappa) = B\frac{2\pi\sin\theta}{q|\sigma|},$$

where θ is the angle between σ and $2\pi i$. Furthermore, equality is attained if and only if the sets $\Pi^{-1}(A_j)$ are straight strips in $\mathbb{C}$ parallel to σ.

Proof. Let $\Pi : \mathbb{C} \to T$ be a universal covering and let ρ be the flat metric $\Pi_*(1/|\sigma|)$ on T. Then ρ is admissible for both T and A_j, $j \in J$. Further $\mathrm{mod}(T,\kappa) = \mathrm{Area}(T,\rho)$ by the standard length-area argument. We get

$$\sum_{j\in J} \mathrm{mod}(A_j) \le \sum_{j\in J} \mathrm{Area}(A_j,\rho) = B\,\mathrm{Area}(T,\rho) = B\frac{2\pi\sin\theta}{q|\sigma|} = B\,\mathrm{mod}(T,\kappa).$$

Equality is attained if and only if the restriction of ρ to each A_j is the extremal (or flat) metric on A_j, that is the last condition of the theorem is fulfilled. □

Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map. Suppose $\alpha \in J_R$ is a repelling k-periodic point with multiplier $\lambda \in \mathbb{C}\setminus\bar{\mathbb{D}}$ and rotation number p/q. Let $\psi : \mathbb{D}(r) \to \Delta(r)$ be a local linearizing parameter for R^k at α. Further let $\Pi_R : \Delta^*(r) \to T_\lambda$ denote the projection given by $\Pi_R := \Pi_\lambda \circ \psi^{-1}$, where $\Delta^*(r) := \Delta(r)\setminus\{\alpha\}$, so that $\Pi_R \circ R^k = \Pi_R$.

Suppose $(\alpha,[\gamma])$ is a periodic access in the Fatou domain Λ. Let U be the unique connected component of $\Delta(r)\cap\Lambda$, which contains the germ of γ, i.e. $\gamma(]0,\epsilon])$ for some $\epsilon > 0$ and let $V := \bigcup_{m\ge 0} \lambda^{qm}\psi^{-1}(U)$. Then $\lambda^q V = V$, since $U \subset R^{kq}(U)$.

LEMMA 4.2. *Let $(\alpha,[\gamma_0])$ and $(\alpha,[\gamma_1])$ be two distinct periodic accesses to α. Let U_0, U_1 and V_0, V_1 be the corresponding sets as above. Then $U_0\cap U_1 = \emptyset$ and $V_0 \cap V_1 = \emptyset$.*

Proof. If $U_0 \cap U_1 \ne \emptyset$, then $U_0 = U_1$, because they are both connected components of the same set. Let $\kappa_0 : [0,1] \to U_0$ be a curve with $\kappa_0(j) \in \gamma_j$ for $j = 0, 1$. For $m \in \mathbb{N}$ define $\kappa_m : [0,1] \to U_0$ by $\kappa_j = R^{-kqm}(\kappa_0)$, where we take the branch of R^{-kqm}, which fixes α. Then the κ_m connects γ_0 and γ_1 in U_0 and converges to α contradicting that γ_0 and γ_1 define distinct accesses. Next assume that $V_0 \cap V_1 \ne \emptyset$ and let $z \in V_0 \cap V_1$. Then $\psi(z/\lambda^{-qm}) \in U_0 \cap U_1$, for some m sufficiently large, because $V_j = \cup_{m\ge 0}\lambda^{qm}\psi^{-1}(U_j)$ and $V_j = V_j/\lambda^q$. But then $U_0 \cap U_1 \ne \emptyset$ contrary to the first part of the Lemma. □

Let U' be the union of the q connected components of $\Delta(r) \cap F_R$ containing the q germs of the curves in the p/q-cycle of γ. Let $\lambda' > 1$ be the conjugate multiplier corresponding to the p/q-cycle of $(\alpha,[\gamma])$ and define $A := \Pi_R(U) = \Pi_R(U') = \Pi_\lambda(V) \subseteq T_\lambda$.

PROPOSITION 4.3. *The set A is an annulus in T_λ of modulus $\pi/\log\lambda'$ and the set U' has a logarithmic density at α equal to the relative flat area of A in T_λ, in particular the opening area of the p/q-cycle of $(\alpha,[\gamma])$ is well defined.*

Proof. We have $A = \Pi_\lambda(V) \simeq V/(z \mapsto \lambda^q z)$, because $\lambda^q V = V$ and $(\lambda^j V) \cap V = \emptyset$ for $j = 1,\ldots,q-1$ by Lemma 4.2. Let $\phi : \Lambda \to \mathbb{D}$ be a Riemann map and let

R_Λ be the conjugate map of R^n on Λ, where n is the period of Λ. Then λ' is the multiplier of the repelling, k'-periodic point $\alpha' \in \mathbb{S}^1$ corresponding to the access $(\alpha, [\gamma])$. Let $\psi' : \mathbb{D}(t) \to \Delta'(t)$ be a local linearizing parameter for $R_\Lambda^{k'}$ at α' with $\psi'(\mathbb{H}_+ \cap \mathbb{D}(t)) \subseteq \mathbb{D}$. We can suppose $\phi(U) \subseteq \Delta'(t)$, decreasing r if necessary. Define $\Psi : \psi^{-1}(U) \to \mathbb{D}(t) \cap \mathbb{H}_+$ by $\Psi = (\psi')^{-1} \circ \phi \circ \psi$. Then Ψ is univalent and satisfies the relation $\Psi(z/\lambda^q) = \Psi(z)/\lambda'$. Thus Ψ extends to a univalent map from V into $\mathbb{H}_+$. We shall prove that Ψ is onto. Let $w \in \mathbb{H}_+$, it suffices to find $m \geq 0$ such that $w(\lambda')^{-m} \in \Psi(V)$, because of the relation $\Psi(z/\lambda^q) = \Psi(z)/\lambda'$. Likewise we can suppose $|w| < t$. Let $\hat{\gamma}_1 :]0, |w|] \to \mathbb{H}_+$ be the arc $\hat{\gamma}_1(t) = tw/|w|$. Then the arc $\gamma_1 := \phi \circ \psi' \circ \hat{\gamma}_1$ converges to α, when the parameter tends to 0 (see the proof of Proposition 2.1). Thus it defines an access to α in Λ. Evidently $\gamma_1 \in [\gamma]$, as $\gamma(t) = \phi \circ \psi'(t)$ is the center curve of its access. In particular $\gamma_1(t) \in U$ for t small and thus $w(\lambda')^{-m} = \hat{\gamma}_1(|w|(\lambda')^{-m}) \in \Psi(\psi^{-1}(U)) \subset \Psi(V)$ for $m \in \mathbb{N}$ big. Let A' be the annulus $\mathbb{H}_+/(z \mapsto \lambda' z)$. Then Ψ descends to a biholomorphic map between A and A'. In particular A is an annulus and a simple calculation, which we leave to the reader, shows that $\mathrm{mod}(A) = \mathrm{mod}(A') = \pi/\log\lambda'$.

To prove the second part of the Proposition, let B denote the relative flat area of the annulus A in T_λ. Note that the set $V' = \cup_{j=0}^{q-1}(\lambda^j V)$ has logarithmic density B at 0, because $V' = \Pi_\lambda^{-1}(A)$ and $(\Pi_\lambda)_*(1/|z|)$ is a flat metric on T_λ. Let $W' = \psi^{-1}(U') \subseteq V' \cap \mathbb{D}(r)$. It suffices to prove that W' has logarithmic density B at 0, by conformal invariance of logarithmic density.

Let $\epsilon > 0$ be given, we can choose a compact subset K of A such that the relative flat area of K is greater than $B - \epsilon$. There exists a compact subset $\hat{K}$ of W' with $\Pi_\lambda(\hat{K}) \supseteq K$, because K is a compact subset of $A = \Pi_\lambda(W')$ and W' is open. Let $t = \min\{|z| | z \in \hat{K}\}$, then $\Pi_\lambda(W' \cap \overline{A(s/|\lambda|, s)}) \supset K$ for all $0 < s \leq t$, because $A(s/|\lambda|, s)$ is a fundamental domain for Π_λ and $W'/\lambda \subset W'$. For $0 < \delta < t$ write $\log t/\delta = n \log|\lambda| + \delta'$ with $0 \leq \delta' < \log|\lambda|$. This gives us the estimate

$$\frac{\mathrm{Area}\left((V' \cap A(\delta, r)), \frac{1}{|z|}\right)}{\mathrm{Area}\left(A(\delta, r), \frac{1}{|z|}\right)} \geq \frac{\mathrm{Area}\left((W' \cap A(\delta, r)), \frac{1}{|z|}\right)}{\mathrm{Area}\left(A(\delta, r), \frac{1}{|z|}\right)}$$

$$\geq \frac{\mathrm{Area}\left((W' \cap A(t/|\lambda|^n, t)), \frac{1}{|z|}\right)}{\mathrm{Area}\left(A(\delta, r), \frac{1}{|z|}\right)}$$

$$\geq \frac{2\pi n \log|\lambda|(B - \epsilon)}{2\pi(\log t/\delta + \log r/t)} \geq \frac{B - \epsilon}{1 + \dfrac{\log t/r}{\log t/\delta}}$$

as $W' \subset V'$. Since the first term tends to B and the last term tends to $B - \epsilon$, when δ tends to 0, and $\epsilon > 0$ was arbitrary, we also deduce that the second term converges to B. That is W' has logarithmic density B at 0 and so has U' at α. □

Let $\kappa := \Pi_R(\gamma)$, then κ is a non-trivial Jordan curve in T_λ and the proof of Proposition 4.3 above shows that κ is the unique closed hyperbolic geodesic of the annulus A (γ

is the center curve of its access), in particular A is homotopic to κ. Also the rotation number of κ equals the rotation number p/q of α.

Proof of Theorem C. As above let α be a repelling periodic point for R with rotation number p/q, multiplier $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$ and period $k \in \mathbb{N}$. Suppose $\lambda_1, \dots, \lambda_N$ are conjugate multipliers corresponding to distinct p/q-cycles of periodic accesses to α and let $B_1, \dots, B_N$ be the opening areas of the same p/q-cycles. Let $(\alpha, [\gamma_1]), \dots, (\alpha, [\gamma_N])$ be periodic accesses to α representing the p/q-cycles and let $A_1, \dots, A_N$ be the annuli corresponding as above to the respective p/q-cycles of accesses. Lemma 4.2 implies that the annuli are disjoint and thus homotopic to say $\kappa = \Pi_R(\gamma_1)$ by the remark following Proposition 4.3. Since the rotation numbers of α and κ are equal, there exists a logarithm L of λ such that the associated segment of κ is $qL - p2\pi i$. Let $B = \sum_{j=1}^N B_j$, then we get from Lemma 4.1

$$\sum_{j=1}^{N} \frac{\pi}{\log \lambda_j} \leq B \frac{2\pi \sin\theta}{q|qL - p2\pi i|},$$

where θ is the angle between $qL - p2\pi i$ and $2\pi i$. This inequality is equivalent to the inequality in Theorem C. □

Proof of Theorem D. Let α be a repelling k-periodic point, with rotation number p/q and multiplier $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$. Suppose further that α is periodically accessible from Λ. Let $\lambda_1, \dots, \lambda_N$ be the conjugate multipliers of the p/q-cycles of accesses to α represented in Λ and let R_Λ be a conjugate map on Λ. The map R_Λ is a finite Blaschke product, in particular it is holomorphic in a neighbourhood of the compact set $\mathbb{S}^1$. Let $M = \max\{|R'_\Lambda(z)| | z \in \mathbb{S}^1\}$. Then for each $j = 1, \dots N$, $\lambda_j \leq M^{kq/n}$, where n is the period of Λ. Let $M_\Lambda = \log M$, then Theorem C implies that there exists a logarithm L of λ such that

$$\left|L - \frac{p}{q}2\pi i\right| \leq \frac{2\sin\theta}{q^2 \sum_{j=1}^N \frac{\pi}{\log \lambda_j}} \leq \frac{2k \sin\theta}{qnN} M_\Lambda,$$

where θ is the angle between $L - (p/q)2\pi i$ and $2\pi i$. This proves Theorem D. To prove the bound on M_Λ, when Λ is hyperbolic, we can suppose that 0 is an attracting fixed point for R_Λ. Then

$$R_\Lambda = \sigma z^k \prod_{j=1}^{d-k} \frac{z - a_j}{1 - \overline{a_j} z}$$

where $\sigma \in \mathbb{S}^1$, d is the degree of R_Λ, $1 \leq k$ is the degree of 0 as a fixed point and the a_j are the pre-images of 0 distinct from 0 itself. A direct calculation shows that

$$|R'_\Lambda(z)| \leq k + \sum_{j=1}^{d-k} \frac{1 + |a_j|}{1 - |a_j|},$$

let $r = \max\{|a_j| | j = 1, \dots, d-k\}$ then

$$M_\Lambda \leq \log |R'_\Lambda(z)| \leq \log d + \log \frac{1+r}{1-r} = \log d + D,$$

where $D := \log((1+r)/(1-r))$. This completes the proof, since D is the maximal hyperbolic distance in $\mathbb{D}$ between 0 and its pre-images by R_Λ, and ϕ is a hyperbolic isometry. □

Corollary D.1 is an immediate consequence of Corollary B.1 and Theorem D, because ∞ is the only pre-image of ∞.

Before we prove Corollary D.2 we need to introduce *the external map* of a polynomial-like mapping and we need some results in hyperbolic geometry. The external map shall play the role of 'flat' model for the dynamics on $U'\backslash K_f$, similar to the role played by the disc model in the simply connected cases. The external map already plays a fundamental role in the paper of Douady and Hubbard. We will however recall the construction in the case of connected filled-in Julia set, which is what we need.

Let $f : U' \to U$ be a polynomial-like mapping with connected filled-in Julia set and with $\mathrm{mod}(U\backslash K_f) = m$. Let

$$\phi : U\backslash K_f \to W_+ := \{z \in \mathbb{C} | 1 < |z| < \exp(2\pi m)\}$$

be a conformal equivalence with $|\phi(z)| \to 1$ as $\mathrm{dist}(z, K_f) \to 0$. Set $W'_+ := \phi(U'\backslash K_f)$ and define $h_+ : W'_+ \to W_+$ by $h_+(z) := \phi \circ f \circ \phi^{-1}(z)$. Then h_+ is a holomorphic d-fold covering map. Let $\zeta : z \mapsto 1/\bar{z}$ denote the inversion with respect to the unit circle $\mathbb{S}^1$. Set $W_- := \zeta(W_+)$, $W'_- := \zeta(W'_+)$ and let $W := W_+ \cup W_- \cup \mathbb{S}^1$ and $W' := W'_+ \cup W'_- \cup \mathbb{S}^1$. By the Schwarz reflection principle h_+ extends to a holomorphic, expanding d-fold covering map, the full external map, $h : W' \to W$ with $h(\mathbb{S}^1) = \mathbb{S}^1 = h^{-1}(\mathbb{S}^1)$. The restriction $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a real analytic d-fold covering map of $\mathbb{S}^1$, called the external map or external class of the polynomial like mapping (U, U', f). It satisfies $|h'(z)| > 1$, $\forall z \in \mathbb{S}^1$.

Let $\lambda_{\mathbb{D}} = 2/(1 - |z|^2)$ and $d_{\mathbb{D}}(\cdot, \cdot)$ denote the hyperbolic metric, respectively the hyperbolic distance in $\mathbb{D}$. Likewise for an open subset $U \subset \mathbb{C}$ isomorphic to $\mathbb{D}$ let λ_U and $d_U(\cdot, \cdot)$ denote the hyperbolic metric, respectively the hyperbolic distance on U.

LEMMA 4.4. *Let $U \subseteq \mathbb{C}$ be an open subset conformally equivalent to $\mathbb{D}$. Then $\forall R \in \mathbb{R}_+$ and $\forall z_1, z_2 \in U$ with $d_U(z_1, z_2) \leq R$*

$$\exp(-2R) \leq \frac{\lambda_U(z_1)}{\lambda_U(z_2)} \leq \exp(2R).$$

Proof. Let $z_1, z_2 \in U$ satisfy $d_U(z_1, z_2) \leq R$. Let $\phi : \mathbb{D} \to U$ be a conformal equivalence with $\phi(0) = z_1$, and let $x := \phi^{-1}(z_2)$. Then ϕ' satisfies the distortion theorem for univalent functions on the unit disc, i.e.

$$\forall z \in \mathbb{D} : \frac{1 - |z|}{(1 + |z|)^3} \leq \left|\frac{\phi'(z)}{\phi'(0)}\right| \leq \frac{1 + |z|}{(1 - |z|)^3}.$$

We evaluate this inequality at x and multiply by $\lambda_{\mathbb{D}}(0)/\lambda_{\mathbb{D}}(x) = 1 - |x|^2$, thus we obtain

$$\frac{(1 - |x|)^2}{(1 + |x|)^2} \leq \left|\frac{\phi'(x) \cdot \lambda_D(0)}{\phi'(0) \cdot \lambda_D(x)}\right| = \left|\frac{\lambda_U(z_1)}{\lambda_U(z_2)}\right| \leq \frac{(1 + |x|)^2}{(1 - |x|)^2}.$$

Since $\log((1 + |x|)/(1 - |x|)) = d_{\mathbb{D}}(0, x) = d_U(z_1, z_2) \leq R$ the lemma follows. □

PROPOSITION 4.5. *Let $h : W' \to W = \{z \in \mathbb{C} | \exp(-2\pi m) < |z| < \exp(2\pi m)\}$ be a d-fold holomorphic covering map with $h(\mathbb{S}^1) = \mathbb{S}^1 = h^{-1}(\mathbb{S}^1)$. Then*

$$\forall z \in \mathbb{S}^1 : |h'(z)| \leq d \exp\left(\frac{d\pi}{2m}\right).$$

Proof. We shall first prove that

$$\forall z_1, z_2 \in \mathbb{S}^1 : \exp\left(-\frac{d\pi}{2m}\right) \leq \left|\frac{h'(z_1)}{h'(z_2)}\right| \leq \exp\left(\frac{d\pi}{2m}\right). \tag{4.1}$$

Let $\widetilde{W}'$ and $\widetilde{W}$ denote the sets $\exp^{-1}(W')$ respective $\exp^{-1}(W)$. Let $\tilde{h} : \widetilde{W}' \to \widetilde{W}$ denote a lift of h, i.e. $\exp\circ\tilde{h} = h \circ \exp$ on $\widetilde{W}'$. Then $\tilde{h}(w + 2\pi i) = \tilde{h}(w) + d2\pi i$ and $\tilde{h}$ is a conformal equivalence, in particular it is a hyperbolic isometry, and the respective hyperbolic metrics satisfy $\lambda_{\widetilde{W}'}(w) = \lambda_{\widetilde{W}}(\tilde{h}(w))|\tilde{h}'(w)|$. Since $\widetilde{W} = \{w \in \mathbb{C}|\ |\Re(w)| < 2\pi m\}$ we have $\lambda_{\widetilde{W}}(w) = 1/4m$ for all $w \in i\Re$. Thus for any pair of points $z_1, z_2 \in \mathbb{S}^1$, there exist $w_1, w_2 \in i\mathbb{R}$ with $d_{\widetilde{W}'}(w_1, w_2) = d_{\widetilde{W}}(\tilde{h}(w_1), \tilde{h}(w_2)) \leq d\pi/4m$ and $\exp(w_j) = z_j$ for $j = 1, 2$. This estimate together with Lemma 4.4 gives

$$\exp\left(-\frac{d\pi}{2m}\right) \leq \left|\frac{\tilde{h}'(w_1)}{\tilde{h}'(w_2)}\right| = \left|\frac{h'(z_1)}{h'(z_2)}\right| \leq \exp\left(\frac{d\pi}{2m}\right) \tag{4.2}$$

since $\tilde{h}'(w) = |h'(\exp(w))|$ on $i\mathbb{R}$. This proves (4.1). The integral over $\mathbb{S}^1$ of $|h'|$ equals $d2\pi$, as $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a d-fold covering map. Thus $\inf\{|h'(z)||z \in \mathbb{S}^1\} \leq d$, which combined with (4.1) yields the Proposition. □

Proof of Corollary D.2. Let $f : U' \to U$ be a polynomial-like mapping of degree d, with K_f connected. Any repelling periodic point for f is periodically accessible from the 'Fatou domain' $U\backslash K_f$, by Corollary B.3. Let α be a repelling, k-periodic point for f with multiplier $\lambda \in \mathbb{C}\backslash\bar{\mathbb{D}}$ and rotation number p/q. We may repeat the arguments leading to Theorem D, replacing R by f, Λ by $U\backslash K_f$ and R_Λ by the external map h of f, to obtain a logarithm L of λ such that

$$|L - (p/q)2\pi i| \leq \frac{2k\sin\theta}{qN}\log M,$$

where θ is the angle between $L - (p/q)2\pi i$ and $2\pi i$, N is the number of cycles of periodic accesses $(\alpha, [\gamma])$ in $U\backslash K_f$ and $M = \max\{|h'(z)||z \in \mathbb{S}^1\}$. If we let m denote the modulus of the annulus $U\backslash K_f$, then Proposition 4.5 applied to the full external map of f shows that $M \leq d\exp(d\pi/2m)$. This completes the proof of Corollary D.2. □

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